Splits and Tight Spans of Convex Polytopes

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte Dissertation

von

Dipl.-Math. Sven Herrmann
aus Darmstadt

Referent:	Prof. Dr. Michael Joswig
Koreferenten:	Prof. Dr. Jan Hendrik Bruinier
	Prof. Dr. Vincent Moulton
Tag der Einreichung:	17. Dezember 2008
Tag der mündlichen Prüfung:	3. Februar 2009

Darmstadt 2009
D 17

Deutsche Zusammenfassung

Eine polytopale Unterteilung eines konvexen Polytops P ist eine Menge von Polytopen deren Ecken Ecken von P sind, deren Vereinigung P ergibt und die eine gewisse Schnittbedingung erfüllen. Polytopale Unterteilungen und insbesondere Triangulierungen, das heißt Unterteilungen in Simplexe, tauchen in den unterschiedlichsten Teilgebieten der Mathematik auf. Eine spezielle Art von Unterteilungen sind die regulären Unterteilungen, die durch Gewichtsfunktionen auf den Ecken von P definiert werden: Es existiert ein Polytop, das Sekundärpolytop von P, dessen Ecken genau den regulären Triangulierungen entsprechen und dessen Seitenverband isomorph zum Verband aller regulären Unterteilungen von P ist. Die Facetten dieses Polytops entsprechen dann den gröbsten Unterteilungen von P. Die Untersuchung dieser Unterteilungen ist ein Hauptthema dieser Arbeit.

Die einfachst möglichen (nichttrivialen) Unterteilungen von P sind jene mit nur zwei maximalen Seiten, die sogenannten Splits. Ein solcher Split wird von einer Hyperebene definiert, die keine Kanten von P zerschneidet. Ursprünglich stammt diese Begriff aus der Theorie der endlichen metrischen Räume. Dort definiert man den Tight-Span eines metrischen Raumes, der als polytopaler Komplex aufgefasst werden kann, und untersucht die Zerlegung von Tight-Spans und endlichen metrischen Räumen in Split-Metriken. Das sind (Pseudo-)Metriken, die entstehen, in dem man die Grundmenge des Raumes in zwei Teile teilt und den Abstand zwischen Elementen aus verschiedenen Teilen mit 1 und sonst mit 0 definiert. Zwischen endlichen metrischen Räumen und polytopalen Unterteilungen besteht der folgende Zusammenhang: Benutzt man die Werte der Metrik als Gewichtsfunktionen für die Ecken eines bestimmten Polytops, des zweiten Hypersimplex $\Delta(2, n)$, erhält man eine polytopale Unterteilung von $\Delta(2, n)$, deren Komplex der inneren Seiten isomorph zum Tight-Span der Metrik ist. Interessanterweise entspricht dann eine Split-Metrik genau einem Split von $\Delta(2, n)$. In dieser Arbeit werden verschiedene Resultate der Theorie der Tight-Spans und Splits endlicher metrischer Räume auf beliebige Polytope verallgemeinert. Außerdem werden die Polytope klassifiziert, für die alle Triangulierungen durch das Verfeinern von Splits gewonnen werden können.

Neben den Splits werden auch weitere Facetten des Sekundärpolytops untersucht. Hierbei zeigt sich, dass der Tight-Span, der bei Splits ein 2-Simplex ist, also aus einer einzelnen Kante besteht, bei allgemeinen gröbsten Unterteilungen sehr kompliziert werden kann. Dies führt zur Definition einer speziellen Klasse gröbster Unterteilungen, den k-Splits. Diese kommen den Splits in ihren Eigenschaften am nächsten, insbesondere ist ihr Tight-Span ein $(k-1)$-dimensionaler Simplex.

Weitere Objekte, die in dieser Dissertation untersucht werden, sind die (tropische) Grassmann-Varietät $\mathrm{Gr}(k, n)$, die Tropikalisierung der Grassmann-Mannigfaltigkeit, welche der Raum aller k-dimensionalen Unterräume eines Vektorraums ist, und eine enge

Verwandte, die Dress-Prävarietät $\mathrm{Dr}(k, n)$. Diese kann äquivalent definiert werden als der Raum aller Gewichtsfunktionen des Hypersimplex $\Delta(k, n)$ (einer Verallgemeinerung des zweiten Hypersimplex), die sogenannte Matroid-Unterteilungen definieren, oder als der Raum aller tropischen Plücker-Vektoren. Diese beiden Betrachtungsweisen definieren unterschiedliche Fächer-Strukturen, von denen im Fall $k = 3$ gezeigt wird, dass sie übereinstimmen. Des Weiteren wird gezeigt, dass der Komplex aller Unterteilungen von $\Delta(k, n)$, die Verfeinerungen von Splits sind, deren definierende Hyperebenen sich nicht im Inneren von $\Delta(k, n)$ schneiden, ein Unterkomplex der Dress-Prävarietät ist. Außerdem werden die Räume $\mathrm{Dr}(3, 7)$ und $\mathrm{Gr}(3, 7)$ explizit berechenet.

Acknowledgments

I would like to express my gratitude to my adviser Michael Joswig for his continuous support during the work on my thesis. Further thanks go to my other coauthors Anders Jensen and Bernd Sturmfels, and to Jesús de Loera and the UC Davis for hosting me in fall 2007. I thank Andrea Peter and Lars Schewe for proof-reading parts of the manuscript.

Moreover, I would like to thank my family, my friends, and my colleagues for the encouragement throughout the last years. The financial support of the TU Darmstadt via a graduate grant is gratefully acknowledged.

Contents

CHAPTER 1

Introduction and Summary

A *polytopal subdivision* of a convex polytope P is a collection of polytopes with union P such that all vertices of these polytopes are vertices of P and satisfying a certain intersection property. Subdivisions and especially triangulations (i.e., subdivisions into simplices) occur in various parts of mathematics; for an overview see the first chapter of the forthcoming book by de Loera, Rambau, and Santos [**19**]. One way to construct polytopal subdivisions of P is the following: Let $w : \operatorname{Vert} P \to \mathbb{R}$ be a function assigning a weight to each vertex of P. If we now lift each vertex according to its weight and project the lower faces of the resulting polytope down to P, we obtain a subdivision of P. Such subdivisions are called *regular*. It is an important structural result of Gel′fand, Kapranov, and Zelevinsky [**40**] (see also [**39**, Chapter 7]) that there exists a polytope $\operatorname{SecPoly}(P)$, called the *secondary polytope* of P, whose vertices are exactly the regular triangulations of P. Moreover, they showed that the face poset of $\operatorname{SecPoly}(P)$ is isomorphic to the poset of all regular subdivisions of P ordered by refinement. In this way, the facets of $\operatorname{SecPoly}(P)$ correspond to those regular subdivisions of P that can only be coarsened by the trivial subdivision. The main aim of this thesis is to begin an investigation of these coarsest subdivisions.

The simplest possible (non-trivial) subdivisions of a polytope one can think of are those with exactly two maximal faces. These are called *splits* and can be obtained by cutting P with any hyperplane not cutting through any edge of P. The original context in that splits appeared is the following: Consider a finite metric space, that is, a function $d : \binom{[n]}{2} \to \mathbb{R}_{\geq 0}$ satisfying the triangle inequality. The *tight span* $\mathcal{T}(d)$ of d is the set of all $x \in \mathbb{R}^n$ satisfying $x_i = \sup_{i \neq j}(d(i, j) - x_j)$. This set $\mathcal{T}(d)$ can also be considered as the complex of bounded faces of a certain polyhedron and hence has a natural structure as a polytopal complex. This construction goes back to Isbell [**55**] (as the *injective hull* of a metric space) and was rediscovered by Dress [**23**], and Chrobak and Lamore [**17, 16**] in two completely different contexts, motivated by applications in biology (phylogenetic analysis) and computer science (k-server problem), respectively. A main ingredient in the study of tight spans of finite metric spaces is the notion of *splits*. These are (pseudo-) metrics obtained by partitioning the ground set into two parts and defining the distance between elements in different parts to be 1 and 0 else. Tight spans of splits are line segments, which are the simplest non-trivial polyhedral complexes. There has been a lot of study on the splits of tight span and finite metric spaces by Dress and his collaborators (see e.g., [**4, 24, 25, 27**]).

The crucial observation linking the theory of finite metric spaces with subdivisions of polytopes is the following (see Sturmfels and Yu [**91**]): The tight span of a metric on n points is dual to the complex of interior faces of the subdivision of a certain polytope, the second hypersimplex $\Delta(2, n)$. In view of this observation, we will generalize the notion of tight spans to arbitrary polytopes and study the facets of $\mathrm{SecPoly}(P)$ in terms of their possible tight spans. It turns out that tight spans of general regular subdivisions can be fairly arbitrary; but there are several restrictions on the tight spans of facets of the secondary polytope.

Not that obvious is the fact that also the notion of a split can be generalized to general polytopes leading to the notion of a split of a convex polytope defined above. This allows us to extend several results for splits of metrics to splits of convex polytopes. The most important one is the Split Decomposition Theorem, which states that an arbitrary weight function of a polytope P can be canonically decomposed into a sum of split weight functions together with a split prime remainder. This result was also obtained earlier by Hirai [**50**], who generalized the notion of split in a slightly different context.

Given a set $\mathcal{S}$ of splits, we define it to be *compatible* if the hyperplanes that define the splits do not meet in the interior of P and *weakly compatible* if the common refinement of all splits in $\mathcal{S}$ is a valid subdivision of P without new interior points. The set of all compatible splits gives rise to a simplicial complex, the split complex of P. We will give a complete description of the compatibility relation of hypersimplices $\Delta(k, n)$, the convex hull of all $0/1$-vectors of length n with exactly k ones.

This general discussion about splits and tight spans will be carried out in Chapters 2* and 3. Despite the results already mentioned, we will give examples of splits and compatibility relations for several polytopes, generalize the notion of splits to oriented matroids, and discuss the arrangements of split hyperplanes. As an application to tropical geometry, we will show that the split complex of $\Delta(k, n)$ is a subcomplex of the Dressian $\mathrm{Dr}(k, n)$ (see below).

In Chapter 4†, we will study *totally splittable polytopes*, those polytopes for which all triangulations can be obtained as refinements of splits. It turns out that a complete classification is possible, and that the classes of polytopes obtained are exactly those for which the secondary polytope was still known.

In Chapter 5, the theory of splits and tight spans is generalized from polytopes to arbitrary point configurations. Most results can be also be obtained in this more general setting, although at some points a bit more care is needed. We will further show that any tight span occurring for a point configuration occurs for some polytope, too, and that each polytope can be the tight span of some subdivision of another polytope.

Motivated by our results about splits, we start studying other facets of $\mathrm{SecPoly}(P)$ in Chapter 6. If such a facet of $\mathrm{SecPoly}(P)$ has exactly k maximal cells, it is called a k-subdivision. We will examine the tight spans of k-subdivisions for small k and give some general necessary conditions for a polytopal complex that can appear as a tight span of a k-subdivision. It turns out that the tight spans of general k-subdivisions can be very complicated. Therefore, we define as a special class of k-subdivisions those whose tight spans are simplices (of dimension $k-1$). These are called k-splits. Since, in this notion, splits are precisely 2-splits, we have a direct generalization of splits. A thorough investigation of k-splits, even for fixed small $k \geq 3$, is beyond the scope of this

*Chapter 2 is joint work with Michael Joswig, published in Münster Journal of Mathematics [**47**].
†Chapter 4 is joint work with Michael Joswig [**48**].

thesis; however, we believe that this would help a lot in understanding the structure of secondary polytopes. For example, a classification of totally 3-splittable polytopes, as it was done for totally splittable polytopes in Chapter 4, could lead to new interesting classes of polytopes whose secondary polytope might be described explicitly.

The main topic of Chapter 7[‡] is the investigation of two special tropical varieties, the tropical Grassmannian and the Dressian. The *tropical Grassmannian* $\mathrm{Gr}(k,n)$ of Speyer and Sturmfels [**84**] is the space of all $(k-1)$-dimensional tropical linear spaces in tropical $(n-1)$-dimensional space, so it is the tropical analogue of the Grassmannian. The *Dressian* $\mathrm{Dr}(k,n)$ is the space of tropical Plücker vectors. It is closely related to the tropical Grassmannian and can also be described as the space of all weight vectors of $\Delta(k,n)$ that introduce so-called *matroid subdivisions* on $\Delta(k,n)$. These two description give rise to two different fan structures. We will show that these structures agree for $k = 3$, and that a matroid subdivision of $\Delta(3,n)$ uniquely corresponds to an arrangement of n metric trees. Furthermore, we explicitly compute $\mathrm{Gr}(3,7)$ and $\mathrm{Dr}(3,7)$ and give descriptions of the bounded parts of all combinatorial types of tropical planes in 6-space. These bounded parts are special tight spans of $\Delta(k,n)$. Whereas the dimension of $\mathrm{Gr}(k,n)$ equals the dimension of the ordinary Grassmannian, which is well known, the dimension of $\mathrm{Dr}(k,n)$ is not known in general. For the case $k = 3$, we will show that $\dim(\mathrm{Dr}(3,n))$ is of order n^2, much bigger than $\dim(\mathrm{Gr}(3,n))$, which is linear in n.

The thesis is concluded with a chapter on further topics and research directions in connection with splits and subdivisions of polytopes. The topics are Ehrhart theory and commutative algebra, finite metric spaces, quasi split subdivisions, and fiber polytopes.

[‡]Chapter 7 is joint work with Anders Jensen, Michael Joswig, and Bernd Sturmfels [**45**].

CHAPTER 2

Splitting Polytopes

This chapter is joint work with Michael Joswig, published in Münster Journal of Mathematics [**47**].

2.1. Introduction

A real-valued weight function w on the vertices of a polytope P in $\mathbb{R}^d$ defines a polytopal subdivision of P by way of lifting to $\mathbb{R}^{d+1}$ and projecting the lower hull back to $\mathbb{R}^d$. The set of all weight functions on P has the natural structure of a polyhedral fan, the *secondary fan* $\mathrm{SecFan}(P)$. The rays of $\mathrm{SecFan}(P)$ correspond to the coarsest (regular) subdivisions of P. This chapter deals with the coarsest subdivisions with precisely two maximal cells. These are called *splits*.

Hirai proved in [**50**] that an arbitrary weight function on P admits a canonical decomposition as a linear combination of split weights with a *split prime* remainder. This generalizes a classical result of Bandelt and Dress [**4**] on the decomposition of finite metric spaces, which proved to be useful for applications in phylogenomics; e.g., see Huson and Bryant [**54**]. We give a new proof of Hirai's split decomposition theorem which establishes the connection to the theory of secondary fans developed by Gel′fand, Kapranov, and Zelevinsky [**40**].

Our main contribution is the introduction and the study of the *split complex* of a polytope P. This comes about as the clique complex of the graph defined by a *compatibility* relation on the set of splits of P. A first example is the boundary complex of the polar dual of the $(n-3)$-dimensional associahedron, which is isomorphic to the split complex of an n-gon. A focus of our investigation is on the hypersimplices $\Delta(k,n)$, which are the convex hulls of the $0/1$-vectors of length n with exactly k ones. We classify all splits of the hypersimplices together with their compatibility relation. This describes the split complexes of the hypersimplices.

Tropical geometry is concerned with the *tropicalization* of algebraic varieties. An important class of examples is formed by the *tropical Grassmannians* $\mathrm{Gr}(k,n)$ of Speyer and Sturmfels [**84**], which are the tropicalizations of the ordinary Grassmannians of k-dimensional subspaces of an n-dimensional vector space (over some field). It is a challenge to obtain a complete description of $\mathrm{Gr}(k,n)$ even for most fixed values of k and n. A better behaved close relative of $\mathrm{Gr}(k,n)$ is the *Dressian** $\mathrm{Dr}(k,n)$ arising from tropicalizing the ideal of quadratic Plücker relations. This is a subfan of the secondary fan of $\Delta(k,n)$, and its rays correspond to coarsest subdivisions of $\Delta(k,n)$ whose (maximal) cells are matroid polytopes; see Kapranov [**60**] and Speyer [**83**]. As one of our main

*In the original paper, the Dressian was called *tropical pre-Grassmannian*. See the introduction of Chapter 7 for the motivation of the term Dressian.

results, we prove that the split complex of $\Delta(k,n)$ is a subcomplex of $\mathrm{Dr}'(k,n)$, the intersection of the fan $\mathrm{Dr}(k,n)$ with the unit sphere in $\mathbb{R}^{\binom{n}{k}}$. Moreover, we believe that our approach can be extended further to obtain a deeper understanding of the Dressian; see also Chapter 7.

This chapter is organized as follows. We start out with the investigation of general weight functions of a polytope P and their coherence. Two weight functions are *coherent* if there is a common refinement of the subdivisions that they induce on P. As an essential technical device for the subsequent sections, we introduce the *coherency index* of two weight functions on P. This generalizes the definition of Koolen and Moulton for $\Delta(2,n)$ [**65**, Section 4.1].

The third section then deals with splits of polytopes and the corresponding weight functions. As a first result we give a concise new proof of the split decomposition theorems of Bandelt and Dress [**4**, Theorem 2] and Hirai [**50**, Theorem 2.2].

A split subdivision of the polytope P is clearly determined by the affine hyperplane spanned by the unique interior cell of codimension one. A set of splits is *compatible* if no two of the corresponding split hyperplanes meet in the (relative) interior of P. The *split complex* $\mathrm{Split}(P)$ is the abstract simplicial complex of compatible sets of splits of P. It is an interesting fact that the subdivision of P induced by a sum of weights corresponding to a compatible system of splits is dual to a tree. In this sense, $\mathrm{Split}(P)$ can always be seen as a “space of trees”.

In Section 2.5, we study the hypersimplices $\Delta(k,n)$. Their splits are classified and explicitly enumerated. Moreover, we characterize the compatible pairs of splits. The purpose of the short Section 2.6 is to specialize our results for arbitrary hypersimplices to the case $k = 2$. A metric on a finite set of n points yields a weight function on $\Delta(2,n)$; and hence all the previous results can be interpreted for finite metric spaces. This is the classical situation studied by Bandelt and Dress [**3, 4**]. Notice that some of their results had already been obtained by Isbell [**55**] much earlier.

Section 2.7 bridges the gap between the split theory of the hypersimplices and matroid theory. This way, as one key result, we can prove that the split complex of the hypersimplex $\Delta(k,n)$ is a subcomplex of the Dressian $\mathrm{Dr}'(k,n)$. We conclude this chapter with a list of open questions.

2.2. Coherency of Weight Functions

Let $P \subset \mathbb{R}^{d+1}$ be a polytope with vertices $v_1,\dots,v_n$. We form the $n\times(d+1)$-matrix V whose rows are the vertices of P. For technical reasons, we make the assumption that P is d-dimensional and that the (column) vector $\mathbb{1} := (1,\dots,1)$ is contained in the linear span of the columns of V. In particular, this implies that P is contained in some affine hyperplane which does not contain the origin. A *weight function* $w : \mathrm{Vert}\,P \to \mathbb{R}$ of P can be written as a vector in $\mathbb{R}^n$. Now each weight function w of P gives rise to the unbounded polyhedron

$$\mathcal{E}_w(P) \;:=\; \left\{x \in \mathbb{R}^{d+1} \;\middle|\; Vx \geq -w\right\},$$

the *envelope* of P with respect to w. We refer to Ziegler [**97**] for details on polytopes.

If w_1 and w_2 are both weight functions of P, then $Vx \geq -w_1$ and $Vy \geq -w_2$ implies $V(x+y) \geq -(w_1+w_2)$. This yields the inclusion

$$\mathcal{E}_{w_1}(P) + \mathcal{E}_{w_2}(P) \;\subseteq\; \mathcal{E}_{w_1+w_2}(P)\,. \tag{2.1}$$

If equality holds in (2.1) then (w_1, w_2) is called a *coherent decomposition* of $w = w_1 + w_2$. (Note that this must not be confused with the notion of "coherent subdivision" which is sometimes used instead of "regular subdivision".)

EXAMPLE 2.1. We consider a hexagon $H \subset \mathbb{R}^3$ whose vertices are the columns of the matrix

$$V^{\mathsf{T}} \;=\; \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix}$$

together with the three weight functions $w_1 = (0,0,1,1,0,0)$, $w_2 = (0,0,0,1,1,0)$, and $w_3 = (0,0,2,3,2,0)$. Again we identify a matrix with the set of its rows. A direct computation then yields that $w_1 + w_2$ is not coherent, but both $w_1 + w_3$ and $w_2 + w_3$ are coherent.

Each *face* of a polyhedron, that is, the intersection with a supporting hyperplane, is again a polyhedron which can be bounded or not. A polyhedron is *pointed* if it does not contain an affine subspace or, equivalently, its lineality space is trivial. This implies that the set of all bounded faces is non-empty and forms a polytopal complex. This polytopal complex is always contractible (see Hirai [**49**, Lemma 4.5]). The polytopal complex of bounded faces of the polyhedron $\mathcal{E}_w(P)$ is called the *tight span* of P with respect to w and is denoted by $\mathcal{T}_w(P)$.

LEMMA 2.2. *Let $w = w_1 + w_2$ be a decomposition of weight functions of P. Then the following statements are equivalent.*

(a) The decomposition (w_1, w_2) is coherent,
(b) $\mathcal{T}_w(P) \subseteq \mathcal{T}_{w_1}(P) + \mathcal{T}_{w_2}(P)$,
(c) $\mathcal{T}_w(P) \subseteq \mathcal{E}_{w_1}(P) + \mathcal{E}_{w_2}(P)$,
(d) each vertex of $\mathcal{T}_w(P)$ can be written as a sum of a vertex of $\mathcal{T}_{w_1}(P)$ and a vertex of $\mathcal{T}_{w_2}(P)$.

For a similar statement in the special case where P is a second hypersimplex (see Section 2.5 below); see Koolen and Moulton [**64**, Lemma 1.2].

PROOF. If (w_1, w_2) is coherent then, by definition, $\mathcal{E}_w(P) = \mathcal{E}_{w_1}(P) + \mathcal{E}_{w_2}(P)$. Each face F of the Minkowski sum of two polyhedra is the Minkowski sum of two faces F_1, F_2, one from each summand. Now F is bounded if and only if F_1 and F_2 are bounded. This proves that (a) implies (b).

Clearly, (b) implies (c). Moreover, (c) implies (d) by the same argument on Minkowski sums as above.

To complete the proof, we have to show that (a) follows from (d). So assume that each vertex of $\mathcal{T}_w(P)$ can be written as a sum of a vertex of $\mathcal{T}_{w_1}(P)$ and a vertex of $\mathcal{T}_{w_2}(P)$, and let $x \in \mathcal{E}_w(P)$. Then x can be written as $x = y + r$ where $y \in \mathcal{T}_w(P)$ and r is a ray of $\mathcal{E}_w(P)$, that is, $z + \lambda r \in \mathcal{E}_w(P)$ for all $z \in \mathcal{E}_w(P)$ and all $\lambda \geq 0$. It follows that $Vr \leq 0$.

By assumption there are vertices y_1 and y_2 of $\mathcal{T}_{w_1}(P)$ and $\mathcal{T}_{w_2}(P)$ such that $y = y_1 + y_2$. Setting $x_1 := y_1 + r$ and $x_2 := y_2$ we have $x = x_1 + x_2$ with $x_2 \in \mathcal{E}_{w_2}(P)$. Computing

$$Vx_1 \;=\; V(y_1 + r) \;\leq\; Vy_1 + Vr \;\leq\; -w_1 + 0 \;=\; -w_1\,,$$

we infer that $x_1 \in \mathcal{E}_{w_1}(P)$. Hence w_1 and w_2 are coherent. □

We recall basic facts about cone polarity. For an arbitrary pointed polyhedron $X \subset \mathbb{R}^{d+1}$ there exists a unique polyhedral cone $C(X) \subset \mathbb{R}^{d+2}$ such that

$$X \;=\; \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; (1, x) \in C(P) \right\}.$$

If X is given in inequality description $X = \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; Ax \geq b \right\}$, one has

$$C(X) \;=\; \left\{ y \in \mathbb{R}^{d+2} \;\middle|\; \begin{pmatrix} 1 & 0 \\ -b & A \end{pmatrix} y \geq 0 \right\}.$$

If X is given in a vertex-ray description $P = \operatorname{conv} V + \operatorname{pos} R$, one has

$$C(X) \;=\; \operatorname{pos} \begin{pmatrix} \mathbb{1} & V \\ 0 & R \end{pmatrix}.$$

For any set $M \subseteq \mathbb{R}^{d+2}$ its cone polar is defined as $M^\circ := \{y \in \mathbb{R}^{d+2} \,|\, \langle x, y\rangle \geq 0 \text{ for all } x \in M\}$. If $C = \operatorname{pos} A$ is a cone, it is easily seen that $C^\circ = \{y \in \mathbb{R}^{d+2} \,|\, Ay \geq 0\}$ and that $(C^\circ)^\circ = C$. The cone C° is called the *polar dual* cone of C. Two polyhedra X and Y are *polar duals* if the corresponding cones $C(X)$ and $C(Y)$ are. The face lattices of dual cones are anti-isomorphic.

For the following our technical assumptions from the beginning come into play. Again let P be a d-polytope in $\mathbb{R}^{d+1}$ such that $\mathbb{1}$ is contained in the column span of the matrix V whose rows are the vertices of P. The standard basis vectors of $\mathbb{R}^{d+1}$ are denoted by $e_1, \ldots, e_{d+1}$.

Proposition 2.3. *The polyhedron $\mathcal{E}_w(P)$ is affinely equivalent to the polar dual of the polyhedron*

$$\mathcal{L}_w(P) := \operatorname{conv}\{v + w(v)e_{d+1} \;|\; v \in \operatorname{Vert} P\} + \mathbb{R}_{\geq 0} e_{d+1}\,.$$

Moreover, the face poset of $\mathcal{T}_w(P)$ is anti-isomorphic to the face poset of the interior lower faces (with respect to the last coordinate) of $\mathcal{L}_w(P)$.

Proof. Note first, that by our assumption that $\mathbb{1}$ is in the column span of V, up to a linear transformation of $\mathbb{R}^{d+1}$, we can assume that $V = (\bar{V}, \mathbb{1})$ for an $n \times d$-matrix $\bar{V}$. This yields

$$C(\mathcal{E}_w(P)) \;=\; \left\{ x \in \mathbb{R}^{d+2} \;\middle|\; \begin{pmatrix} 1 & 0 & 0 \\ w & \bar{V} & \mathbb{1} \end{pmatrix} x \geq 0 \right\}.$$

On the other hand, we have

$$C(\mathcal{L}_w(P)) \;=\; \operatorname{pos} \begin{pmatrix} \mathbb{1} & \bar{V} & w \\ 0 & 0 & 1 \end{pmatrix},$$

which is linearly isomorphic to $\bar{C} = \operatorname{pos}\left(\begin{smallmatrix} w & \mathbb{1} & \bar{V} \\ 1 & 0 & 0 \end{smallmatrix}\right)$ by a coordinate change, so $\mathcal{E}_w(P)$ and $\mathcal{L}_w(P)$ are polar duals, up to linear transformations.

This way we have obtained an anti-isomorphism of the face lattices of $C(\mathcal{E}_w(P))$ and $C(\mathcal{L}_w(P))$. A face F of $\mathcal{E}_w(P)$ is bounded if and only if no generator of $C(\mathcal{E}_w(P))$ with first coordinate equal to zero is smaller then F in the face lattice. In the dual view, this means that the corresponding face F' of $\mathcal{L}_w(P)$ is greater then a facet which is parallel to the last coordinate axis in the face lattice of $C(\mathcal{L}_w(P))$. But this exactly means that F' is a lower face. So the lattice anti-isomorphism of $C(\mathcal{E}_w(P))$ and $C(\mathcal{L}_w(P))$ induces a poset anti-isomorphism between $\mathcal{T}_w(P)$ and the interior lower faces of $\mathcal{L}_w(P)$. □

The lower faces of $\mathcal{L}_w(P)$ (with respect to the last coordinate) are precisely its bounded faces. Being projected back to $\operatorname{aff} P$ in the e_{d+1}-direction, the polytopal complex of bounded faces of $\mathcal{L}_w(P)$ induces a polytopal decomposition $\Sigma_w(P)$ of P. Note that we only allow the vertices of P as vertices of any subdivision of P. A polytopal subdivision which arises in this way is called *regular*. Two weight functions are *equivalent* if they induce the same subdivision. This allows for one more characterization extending Lemma 2.2.

COROLLARY 2.4. *A decomposition $w = w_1 + w_2$ of weight functions of P is coherent if and only if the subdivision $\Sigma_w(P)$ is the common refinement of the subdivisions $\Sigma_{w_1}(P)$ and $\Sigma_{w_2}(P)$.*

PROOF. By Lemma 2.2, the decomposition $w_1 + w_2$ is coherent if and only if each vertex x of $\mathcal{T}_w(P)$ is a sum of a vertex x_1 of $\mathcal{T}_{w_1}(P)$ and a vertex x_2 of $\mathcal{T}_{w_2}(P)$. In terms of the duality proved in Proposition 2.3, the vertex x corresponds to the maximal cell $F_w(x) := \operatorname{conv}\{v \in \operatorname{Vert} P \mid \langle v, x\rangle = -w\}$ of $\Sigma_w(P)$. Similarly, x_1 and x_2 corresponds to the cells $F_{w_1}(x_1)$ and $F_{w_2}(x_2)$ of $\Sigma_{w_1}(P)$ and $\Sigma_{w_2}(P)$, respectively. In fact, we have $F_w(x) = F_{w_1}(x_1) \cap F_{w_2}(x_2)$, and so $\Sigma_w(P)$ is the common refinement of $\Sigma_{w_1}(P)$ and $\Sigma_{w_2}(P)$. The converse follows similarly. □

EXAMPLE 2.5. In Example 2.1, the tight spans of the three weight functions of the hexagon are line segments:

$$\mathcal{T}_{w_1}(H) = [0,(1,-1,0)], \quad \mathcal{T}_{w_2}(H) = [0,(1,0,-1)], \quad \text{and} \quad \mathcal{T}_{w_3}(H) = [0,(1,-1,-1)].$$

REMARK 2.6. Interesting special cases of tight spans include the following. Finite metric spaces (on n points) give rise to weight functions on the second hypersimplex $P = \Delta(2,n)$. In this case, the tight span can be interpreted as a "space" of trees which are candidates to fit the given metric. This has been studied by Bandelt and Dress [**4**], and this is the context in which the name "tight span" was used first. See also Section 2.6 and Section 8.2 below.

If P is a product of two simplices, the tight span of a lifting function gives rise to a *tropical polytope* introduced by Develin and Sturmfels [**21**], the cells in the resulting regular decomposition of P are the *polytropes* of [**59**].

If P spans the affine hyperplane $x_1 = 1$ and if we consider the weight function defined by $w(v) = v_2^2 + v_3^2 + \cdots + v_{d+1}^2$ for each vertex v of P, then the tight span $\mathcal{T}_w(P)$ is isomorphic to the subcomplex of bounded faces of the Voronoi diagram of $\operatorname{Vert} P$. All maximal cells of the Voronoi diagram are unbounded and hence the tight span is at most $(d-1)$-dimensional. The subdivision $\Sigma_w(P)$ is then isomorphic to the Delone decomposition of $\operatorname{Vert} P$.

Let w and w' be weight functions of our polytope P. We want to have a measure that expresses to what extent the pair of weight functions $(w', w - w')$ deviates from coherence (if at all). The *coherency index* of w with respect to w' is defined as

$$\alpha_{w'}^{w} \;:=\; \min_{x \in \operatorname{Vert} \mathcal{E}_w(P)} \left\{ \max_{x' \in \operatorname{Vert} \mathcal{E}_{w'}(P)} \left\{ \min_{v \in V_{w'}(x')} \left\{ \frac{\langle v, x\rangle + w(v)}{\langle v, x'\rangle + w'(v)} \right\} \right\} \right\} , \tag{2.2}$$

where $V_{w'}(x') = \{v \in \operatorname{Vert} P \mid \langle v, x'\rangle \neq -w'(v)\}$. (That is, $V_{w'}(x')$ is the set of vertices of P that are not contained in the cell dual to x.) The name is justified by the following observation, which generalizes Koolen and Moulton [**65**, Theorem 4.1].

Proposition 2.7. *Let w and w' be weight functions of the polytope P. Moreover, let $\lambda \in \mathbb{R}$ and $\tilde{w} := w - \lambda w'$. Then $w = \tilde{w} + \lambda w'$ is coherent if and only if $0 \leq \lambda \leq \alpha_{w'}^{w}$.*

Proof. Assume that $w = \tilde{w} + \lambda w'$ is coherent. By Lemma 2.2, for each vertex x of $\mathcal{E}_w(P)$ there is a vertex x' of $\mathcal{E}_{w'}(P)$ such that $x - \lambda x'$ is a vertex of $\mathcal{E}_{\tilde{w}}(P)$. We arrive at the following sequence of equivalences:

$$\begin{aligned}
x - \lambda x' \in \mathcal{T}_{\tilde{w}}(P) &\iff -w(v) + \lambda w'(v) \;\leq\; \langle v, x - \lambda x'\rangle \quad \text{for all } v \in \operatorname{Vert} P \\
&\iff \lambda(\langle v, x'\rangle + w'(v)) \;\leq\; \langle v, x\rangle + w(v) \quad \text{for all } v \in \operatorname{Vert} P \\
&\iff \lambda \;\leq\; \frac{\langle v, x\rangle + w(v)}{\langle v, x'\rangle + w'(v)} \quad \text{for all } v \in V_{w'}(x') \\
&\iff \lambda \;\leq\; \min_{v \in V_{w'}(x')} \left\{ \frac{\langle v, x\rangle + w(v)}{\langle v, x'\rangle + w'(v)} \right\} .
\end{aligned}$$

For each vertex x of $\mathcal{E}_w(P)$ there must be some vertex x' of $\mathcal{E}_{w'}(P)$ such that these inequalities hold, and this gives the claim. □

Corollary 2.8. *For two weight function w and w' of P we have*

$$\alpha_{w'}^{w} \;=\; \sup\{\lambda \geq 0 \mid (w - \lambda w', \lambda w') \text{ is a coherent decomposition of } w\} .$$

Corollary 2.9. *If w and w' are weight functions, then $\Sigma_w(P) = \Sigma_{w'}(P)$ if and only if $\alpha_{w'}^{w} > 0$ and $\alpha_{w}^{w'} > 0$.*

The set of all regular subdivisions of the convex polytope P is known to have an interesting structure (see [**19**, Chapter 5] for the details): For a weight function $w \in \mathbb{R}^n$ of P we consider the set $S[w] \subset \mathbb{R}^n$ of all weight functions that are equivalent to w, that is,

$$S[w] \;:=\; \{x \in \mathbb{R}^n \mid \Sigma_x(P) = \Sigma_w(P)\} .$$

This set is called the *secondary cone* of P with respect to w. It can be shown (see e.g., [**19**, Corollary 5.2.10]) that $S[w]$ is indeed a polyhedral cone and that the set of all $S[w]$ (for all w) forms a polyhedral fan $\operatorname{SecFan}(P)$, called the *secondary fan* of P.

It is easily verified that $S[0]$ is the set of all (restrictions of) affine linear functions and that it is the lineality space of every cone in the secondary fan. So this fan can be regarded in the quotient space $\mathbb{R}^n/S[0] \cong \mathbb{R}^{n-d-1}$. If there is no chance for confusion, we will identify $w \in \mathbb{R}^n$ and its image in $\mathbb{R}^n/S[0]$. Furthermore, the secondary fan can be cut with the unit sphere to get a (spherical) polytopal complex on the set of rays in the fan. This complex carries the same information as the fan itself and will also be identified with it.

It is a famous result by Gel′fand, Kapranov, and Zelevinsky [**40**, Theorem 1.7] that the secondary fan is the normal fan of a polytope, the *secondary polytope* $\mathrm{SecPoly}(P)$ of P. This polytope admits a realization as the convex hull of the so-called *GKZ-vectors* of all (regular) triangulations of P. The GKZ-vector $x_\Delta \in \mathbb{R}^n$ of a triangulation Σ is defined as $(x_\Sigma)_v := \sum_S \operatorname{vol} S$ for all $v \in \operatorname{Vert} P$, where the sum ranges over all full-dimensional simplices $S \in \Sigma$ which contain v.

A description in terms of inequalities is given by Lee [**68**, Section 17.6, Result 4]: The affine hull of $\mathrm{SecPoly}(P) \subset \mathbb{R}^n$ is given by the $d + 1$ equations

$$\begin{aligned} \sum_{v \in \operatorname{Vert} P} x_v &= d(d+1) \operatorname{vol} P \quad \text{and} \\ \sum_{v \in \operatorname{Vert} P} x_v v &= ((d+1) \operatorname{vol} P) \cdot c_P \,, \end{aligned} \tag{2.3}$$

where c_P denotes the centroid of P and vol the d-dimensional volume in the affine span of P, which we can identify with $\mathbb{R}^d$. The facet defining inequalities of $\mathrm{SecPoly}(P)$ are

$$\sum_{v \in \operatorname{Vert} P} w(v) x_v \;\geq\; (d+1) \sum_{Q \in \Sigma_w(P)} \operatorname{vol} Q \cdot w^\star(c_Q)\,, \tag{2.4}$$

for all coarsest regular subdivisions $\Sigma_w(P)$ defined by a weight w. Here $w^\star : P \mapsto \mathbb{R}$ denotes the piecewise-linear convex function whose graph is given by the lower facets of $\mathcal{L}_w(P)$.

A weight function w such that for all weight functions w' with $\alpha^w_{w'} > 0$ we have $w' = \lambda w$ (in $\mathbb{R}^n / S[0]$) for some $\lambda > 0$ is called *prime*. The set of all prime weight functions for a given polytope P is denoted $\mathcal{W}(P)$. By this we get directly:

PROPOSITION 2.10. *The equivalence classes of prime weights correspond to the extremal rays of the secondary fan (and hence to the coarsest regular subdivisions or, equivalently, to the facets of the secondary polytope).*

The following is a reformulation of the fact that the set of all equivalence classes of weight functions of P forms a fan (the secondary fan of P).

THEOREM 2.11. *Each weight function w on a polytope P can be decomposed into a coherent sum of prime weight functions, that is, there are $p_1, \ldots, p_k \in \mathcal{W}(P)$ such that $w = p_1 + \cdots + p_k$ is a coherent decomposition.*

PROOF. Each weight function w is contained in some cone of the secondary fan of P. Hence there are extremal rays $r_1, \ldots, r_k$ of the secondary cone and positive real numbers $\lambda_1, \ldots, \lambda_k$ such that $w = \lambda_1 r_1 + \cdots + \lambda_k r_k$; by construction, this decomposition is coherent by Lemma 2.2. From Proposition 2.10 we know that $p_i := \lambda_i r_i$ is a prime weight, and the claim follows. □

Note that this decomposition is usually not unique.

2.3. Splits and the Split Decomposition Theorem

A *split* S of a polytope P is a decomposition of P without new vertices which has exactly two maximal cells denoted by S_+ and S_-. As above, we assume that $P \subset \mathbb{R}^{d+1}$ is d-dimensional and that $\operatorname{aff} P$ does not contain the origin. Then the linear span of $S_+ \cap S_-$ is a linear hyperplane H_S, the *split hyperplane* of S with respect to P. Since S does not induce any new vertices, in particular, H_S does not meet any edge of P in its relative interior. Conversely, each hyperplane which separates P and which does not separate any edge defines a split of P. Furthermore, it is easy to see that a hyperplane defines a split of P if and only if it defines a split on all facets of P that it meets in the (relative) interior. (Note: In the sequel, we will frequently write "interior" instead of "relative interior" if there is no chance for confusion.)

The following observation is immediate. Note that it implies that a hyperplane defines a split if and only if its does not separate any edge.

Observation 2.12. *A hyperplane that meets P in its interior is a split hyperplane of P if and only if it intersects each of its facets F in either a split hyperplane of F or in a face of F.*

Remark 2.13. Since the notion of facets and faces of a polytope only depends on the *oriented matroid* of P it follows from Observation 2.12 that the set splits of a polytope only depend on the oriented matroid of P. This is in contrast to the fact that the set of regular triangulations (see below), in general, depends on the specific coordinatization. See also Section 3.5 for a discussion of oriented matroids.

The running theme of this thesis is: If a polytope admits sufficiently many splits, then interesting things happen. However, one should keep in mind that there are many polytopes without a single split; such polytopes are called *unsplittable*.

Remark 2.14. If v is a vertex of P such that all neighbors of v in P are contained in a common hyperplane H_v then H_v defines a split S_v of P. Such a split is called the *vertex split* with respect to v. For instance, if P is simple then each vertex defines a vertex split.

Since polygons are simple polytopes it follows, in particular, that an unsplittable polytope which is not a simplex is at least three-dimensional. An unsplittable 3-polytope has at least six vertices. An example is a three-dimensional cross polytope whose vertices are perturbed into general position.

Proposition 2.15. *Each 2-neighborly polytope is unsplittable.*

Proof. Assume that S is a split of a 2-neighborly polytope P. Recall that this property means that any two vertices of P are joined by an edge. Choose vertices $v \in S_+ \setminus S_-$ and $w \in S_- \setminus S_+$. Then the segment $[v, w]$ is an edge of P which is separated by the split hyperplane H_S. This is a contradiction to the assumption that S was a split of P. □

It is clear that splits yield coarsest subdivisions; but the following lemma says that they even define facets of the secondary polytope.

Lemma 2.16. *Splits are regular.*

PROOF. Let S be a split of P. We have to show that S is induced by a weight function. Let a be a normal vector of the split hyperplane H_S. We define $w_S : \mathrm{Vert}(P) \to \mathbb{R}$ by

$$w_S(v) \;:=\; \begin{cases} |\langle a, v\rangle| & \text{if } v \in S_+ , \\ 0 & \text{if } v \in S_- . \end{cases} \tag{2.5}$$

Note that this function is well-defined since for $v \in H_S = \mathrm{lin}(S_+ \cap S_-)$ we have $av = 0$. It is now obvious that w induces the split S on P. □

EXAMPLE 2.17. In Example 2.1 the three weight functions w_1, w_2, w_3 define splits of the hexagon H.

By specializing Equation (2.4), a facet defining inequality for the split S is given by

$$\sum_{v \in \mathrm{Vert}\, S_+} |\langle a, v\rangle| x_v \;\geq\; |\langle a, c_{S_+}\rangle|\,(d+1)\,\mathrm{vol}\, S_+ . \tag{2.6}$$

Note that a is a normal vector of the split hyperplane H_S as above, and c_{S_+} is the centroid of the polytope $P \cap S_+$. By taking the inequalities (2.6) for all splits S of P together with the equations (2.3), we get an $(n-d-1)$-dimensional polyhedron $\mathrm{SplitPoly}(P)$ which we will call the *split polyhedron* of P. Obviously, we have $\mathrm{SecPoly}(P) \subseteq \mathrm{SplitPoly}(P)$ so the split polyhedron can be seen as an outer "approximation" of the secondary polytope. In fact, by Remark 2.13, $\mathrm{SplitPoly}(P)$ is a common "approximation" for the secondary polytopes of all possible coordinatizations of the oriented matroid of P. If P has sufficiently many splits, the split polyhedron is bounded; in this case, $\mathrm{SplitPoly}(P)$ is called the *split polytope* of P.

One can show that each simple polytope has a bounded split polyhedron. Here we give two examples.

EXAMPLE 2.18. Let P be a an n-gon for $n \geq 4$. Then each pair of non-neighboring vertices defines a split of P. Each triangulation of P is regular and, moreover, a split triangulation.

The secondary polytope of P is the associahedron Assoc_{n-3}, which is a simple polytope of dimension $n-3$. Since the only coarsest subdivisions of P are the splits it follows that the split polytope of P coincides with its secondary polytope.

EXAMPLE 2.19. The 74 triangulations of the regular 3-cube $C_3 = [-1, 1]^3$ are all regular, and 26 of them are induced by splits. The total number of splits is 14: There are eight vertex splits (C being simple) and six splits defined by parallel pairs of diagonals in an opposite pair of cube facets. The secondary polytope of C is a 4-polytope with f-vector $(74, 152, 100, 22)$; see Pfeifle [**75**] for a complete description.

The split polytope of C_3 is neither simplicial nor simple and its f-vector reads $(22, 60, 52, 14)$. A Schlegel diagram is shown in Figure 2.1.

EXAMPLE 2.20. There are nearly 88 million regular triangulations of the 4-cube $C_4 = [-1, 1]^4$ that come in 235,277 equivalence classes. The 4-cube has four different types of splits: The vertex splits, the split obtained by cutting with $H := \{x \mid \sum x_i = 0\}$ (and its images under the symmetry group of the cube), and, finally, two kinds of splits induced by the two kinds of splits of the 3-cube. The split obtained from the vertex split of the 3-cube is the one discussed in [**53**, Example 20 (The missing split)]. See also [**53**] for a complete discussion of the secondary polytope of C_4. Examples of triangulations of the 4-cube that are induced by splits include the first two in [**53**, Example 10 & Figure 3] and the one whose tight span is shown in Figure 2.4.

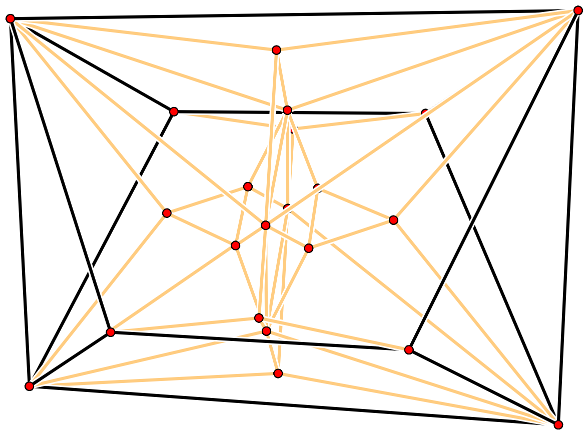

FIGURE 2.1. Schlegel diagram of the split polytope of the regular 3-cube.

A weight function w on a polytope P is called *split prime* if for all splits S of P we have $\alpha^w_{w_S} = 0$. The following can be seen as a generalization of Bandelt and Dress [**4**, Theorem 2], and as a reformulation of Hirai's Theorem 2.2 [**50**].

THEOREM 2.21 (Split Decomposition Theorem). *Each weight function w has a coherent decomposition*

$$w \;=\; w_0 + \sum_{S \text{ split of } P} \lambda_S w_S \,, \tag{2.7}$$

where w_0 is split prime, and this is unique among all coherent decompositions of w.

This is called the *split decomposition* of w.

PROOF. We first consider the special case where the subdivision $\Sigma_w(P)$ induced by w is a common refinement of splits. Then each face F of codimension one in $\Sigma_w(P)$ defines a unique split $S(F)$, namely the one with split hyperplane $H_{S(F)} = \operatorname{lin} F$. Moreover, whenever S is an arbitrary split of P, then $\alpha^w_{w_S} > 0$ if and only if $H_S \cap P$ is a face of $\Sigma_w(P)$ of codimension one. This gives a coherent subdivision $w = \sum_S \alpha^w_{w_S} w_S$, where the sum ranges over all splits S of P. Note that the uniqueness follows from the fact that for each codimension-one-faces of $\Sigma_w(P)$ there is a unique split which coarsens it.

For the general case, we let

$$w_0 \;:=\; w - \sum_{S \text{ split of } P} \alpha^w_{w_S} w_S \,.$$

By construction, w_0 is split prime, and the uniqueness of the split decomposition of w follows from the uniqueness of the split decomposition of $w - w_0$. □

In fact, the sum in (2.7) only runs over all splits in $\mathcal{S}(w) := \{w_S \mid \alpha^w_{w_S} > 0\}$. The uniqueness part of the theorem gives us the following interesting corollary (see also Bandelt and Dress [**4**, Corollary 5], and Hirai [**50**, Proposition 3.6]):

COROLLARY 2.22. *For a weight function w the set $\mathcal{S}(w) \cup \{w_0\}$ is linearly independent. In particular, $|\mathcal{S}(w)| \le n-d-1$, if $|\mathcal{S}(w)| = n-d-1$ then $w_0 = 0$, and if $|\mathcal{S}(w)| = n-d-2$ then w_0 is a prime weight function.*

PROOF. Suppose the set would be linearly dependent. This would yield a relation

$$\sum_{S \in \mathcal{S}} \lambda_S w_S \;=\; \lambda_0 w_0 + \sum_{S \in \mathcal{S}(w) \setminus \mathcal{S}} \lambda_S w_S$$

with coefficients $\lambda_0, \lambda_S \ge 0$ for some $\mathcal{S} \subset \mathcal{S}(w)$. However, this contradicts the uniqueness part of Theorem 2.21 for the weight function $w' := \sum_{S \in \mathcal{S}} \lambda_S w_S$.

The cardinality constraints now follow from the fact that the weight functions live in $\mathbb{R}^n / S[0] \cong \mathbb{R}^{n-d-1}$. □

The next lemma is a specialization of Corollary 2.4 to the case of splits and their weight functions.

LEMMA 2.23. *Let $\mathcal{S}$ be a set of splits for P. Then the following statements are equivalent.*

(a) The corresponding decomposition $w := \sum_{S \in \mathcal{S}} w_S$ is coherent,
(b) there exists a common refinement of all $S \in \mathcal{S}$ (induced by w),
(c) there is a regular triangulation of P which refines all $S \in \mathcal{S}$.

Instead of "set of splits" we equivalently use the term *split system*. A split system is called *weakly compatible* if one of the properties of Lemma 2.23 is satisfied. Moreover, two splits S_1 and S_2 such that $H_{S_1} \cap H_{S_2}$ does not meet P in its interior are called *compatible*. This notion generalizes to arbitrary split systems in different ways: A set $\mathcal{S}$ of splits is called compatible if any two of its splits are compatible. It is *incompatible* any two splits are not compatible, and it is *totally incompatible* if $\bigcap_{S \in \mathcal{S}} H_S$ has a non-empty intersection with the interior of P. It is clear that total incompatibility implies incompatibility, and that compatibility implies weak compatibility (but the converse does not hold, see Example 2.34).

For an arbitrary split system $\mathcal{S}$ we define its weight function as

$$w_{\mathcal{S}} \;:=\; \sum_{S \in \mathcal{S}} w_S \,.$$

If $\mathcal{S}$ is weakly compatible then $\Sigma_{\mathcal{S}}(P) := \Sigma_{w_{\mathcal{S}}}(P)$ is the coarsest subdivision refining all splits in $\mathcal{S}$. We further abbreviate $\mathcal{E}_{\mathcal{S}}(P) := \mathcal{E}_{w_{\mathcal{S}}}(P)$ and $\mathcal{T}_{\mathcal{S}}(P) := \mathcal{T}_{w_{\mathcal{S}}}(P)$.

REMARK 2.24. The split decomposition (2.7) of a weight function w of the d-polytope P can actually be computed using our formula (2.2). Provided we already know the, say, t vertices of the tight span of w and the, say, s splits of P, this takes $O(s\,t\,d\,n)$ arithmetic operations over the reals (or the rationals), where $n = |\operatorname{Vert} P|$.

2.4. Split Complexes and Split Subdivisions

Let P be a fixed d-polytope, and let $\mathcal{S}(P)$ be the set of all splits of P. The notions of compatibility and weak compatibility of splits give rise to two abstract simplicial complexes with vertex set $\mathcal{S}(P)$. We denote them by $\mathrm{Split}(P)$ and $\mathrm{Split}^{\mathrm{w}}(P)$, respectively. Since compatibility implies weak compatibility $\mathrm{Split}(P)$ is a subcomplex of $\mathrm{Split}^{\mathrm{w}}(P)$. Moreover, if $\mathcal{S} \subseteq \mathcal{S}(P)$ is a split system such that any two splits in $\mathcal{S}$ are compatible, then the whole split system $\mathcal{S}$ is compatible. This can also be phrased in graph theory language: The compatibility relation among the splits defines an undirected graph, whose cliques correspond to the faces of $\mathrm{Split}(P)$. In particular, we have the following:

Proposition 2.25. *The split complex* $\mathrm{Split}(P)$ *is a flag simplicial complex.*

Note that we did not assume that P admits any split. If P is unsplittable then the (weak) split complex of P is the void complex $\emptyset$.

Theorem 2.21 tells us that the fan spanned by the rays that induce splits is a simplicial fan contained in (the support of) $\mathrm{SecFan}(P)$. This fan was called the *split fan* of P by Koichi [**63**]. Denoting by $\mathrm{SecFan}'(P)$ the (spherical) polytopal complex which arises from $\mathrm{SecFan}(P)$ by intersecting with the unit sphere, we arrive at the following observation:

Corollary 2.26. *The simplicial complex* $\mathrm{Split}(P)$ *is a subcomplex of the polytopal complex* $\mathrm{SecFan}'(P)$.

Proof. The tight span of a compatible system $\mathcal{S}$ of splits of P is a tree by Proposition 2.30. This implies that the cell C in $\mathrm{SecFan}'(P)$ generated by $\mathcal{S}$ does not contain vertices whose tight span is of dimension greater than one. Thus the vertices of C are precisely the splits in $\mathcal{S}$. □

Remark 2.27. The weak split complex of P is usually not a subcomplex of $\mathrm{SecFan}'(P)$; see Example 2.34. However, one can show that $\mathrm{Split}^{\mathrm{w}}(P)$ is homotopy equivalent to a subcomplex of $\mathrm{SecFan}'(P)$.

From Corollary 2.22 we can trivially derive an upper bound on the dimensions of the split complex and the weak split complex. This bound is sharp for both types of complexes as we will see in Example 2.32 below.

Proposition 2.28. *The dimensions of* $\mathrm{Split}(P)$ *and* $\mathrm{Split}^{\mathrm{w}}(P)$ *are bounded from above by* $n - d - 2$.

A regular subdivision (triangulation) Σ of P is called a *split subdivision (triangulation)* if it is the common refinement of a set $\mathcal{S}$ of splits of P. Necessarily, the split system $\mathcal{S}$ is weakly compatible and a face of $\mathrm{Split}^{\mathrm{w}}(P)$. Conversely, all faces of $\mathrm{Split}^{\mathrm{w}}(P)$ arise in this way.

Corollary 2.29. *If* $\mathcal{S}$ *is a facet of* $\mathrm{Split}^{\mathrm{w}}(P)$ *with* $|\mathcal{S}| = n - d - 1$, *then the split subdivision* $\Sigma_{\mathcal{S}}(P)$ *is a split triangulation.*

Proof. Corollary 2.22 implies that $W := \{w_S \mid S \in \mathcal{S}\}$ is linearly independent and hence a basis of $\mathbb{R}^n/S[0] \cong \mathbb{R}^{n-d-1}$. So the cone spanned by W is full-dimensional and hence corresponds to a vertex of the secondary polytope. □

The following is a characterization of the faces of $\mathrm{Split}(P)$ and says that split complexes are always "spaces of trees".

PROPOSITION 2.30 (Hirai [**50**], Proposition 2.9). *Let $\mathcal{S}$ be a split system on P. Then the following statements are equivalent.*

(a) $\mathcal{S}$ is compatible,
(b) $\mathcal{T}_{\mathcal{S}}(P)$ is one-dimensional, and
(c) $\mathcal{T}_{\mathcal{S}}(P)$ is a tree.

PROOF. Assume that $\Sigma_{\mathcal{S}}(P)$ is induced by the compatible split system $\mathcal{S} \neq \emptyset$. By definition, for any two distinct splits $S_1, S_2 \in \mathcal{S}$ the hyperplanes H_{S_1} and H_{S_2} do not meet in the interior of P. This implies that there are no interior faces in $\Sigma_{\mathcal{S}}(P)$ of codimension greater than one. By Proposition 2.3, this says that $\dim \mathcal{T}_{\mathcal{S}}(P) \leq 1$. Since $\mathcal{S} \neq \emptyset$ we have that $\dim \mathcal{T}_{\mathcal{S}}(P) = 1$. Thus (a) implies (b).

The statement (c) follows from (b) as the tight span is contractible.

Suppose that $\mathcal{T}_{\mathcal{S}}(P)$ is a tree. Then each edge is dual to a split hyperplane. The system $\mathcal{S}$ of all these splits is clearly weakly compatible since it is refined by $\Sigma_{\mathcal{S}}(P)$. Assume that there are splits $S_1, S_2 \in \mathcal{S}$ such that the corresponding split hyperplanes H_{S_1} and H_{S_2} meet in the interior of P. Then $H_{S_1} \cap H_{S_2}$ is an interior face in $\Sigma_{\mathcal{S}}(P)$ of codimension two, contradicting our assumption that $\mathcal{T}_{\mathcal{S}}(P)$ is a tree. This proves (a), and hence the claim follows. □

REMARK 2.31. A d-dimensional polytope is called *stacked* if it has a triangulation without interior faces of dimension less than $d-1$. So it follows from Proposition 2.30 that a polytope is stacked if and only if there exists a split triangulation induced by a compatible system of splits.

EXAMPLE 2.32. Let P be a an n-gon for $n \geq 4$. As already pointed out in Example 2.18, each pair of non-neighboring vertices defines a split of P. Two such splits are compatible if and only if they are weakly compatible.

The secondary polytope of P is the associahedron Assoc_{n-3}, and the split complex of P is isomorphic to the boundary complex of its dual. In particular, $\mathrm{Split}(P) = \mathrm{Split}^{\mathrm{w}}(P)$ is a pure and shellable simplicial complex of dimension $n-4$, which is homeomorphic to $\mathbb{S}^{n-4}$. This shows that the bound in Proposition 2.28 is sharp. From Catalan combinatorics it is known that the (split) triangulations of P correspond to the binary trees on $n-2$ nodes; see [**19**, Section 1.1].

EXAMPLE 2.33. The splits of the regular cross polytope $X_d = \mathrm{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ in $\mathbb{R}^d$ are induced by the d reflection hyperplanes $x_i = 0$. Any $d-1$ of them are weakly compatible and define a triangulation of X_d by Corollary 2.29. (Of course, this can also be seen directly.) All triangulations of X_d arise in this way. This shows that $\mathrm{Split}^{\mathrm{w}}(X_d)$ is isomorphic to the boundary complex of a $(d-1)$-dimensional simplex, which is also the secondary polytope and the split polytope of X_d. Any two reflection hyperplanes meet in the interior of X_d, whence no two splits are compatible. This says that $\mathrm{Split}(X_d)$ consists of d isolated points.

EXAMPLE 2.34. As we already discussed in Example 2.19 the 3-cube $C_3 = [-1,1]^3$ has a total number of 14 splits. The split complex $\mathrm{Split}(C)$ is three-dimensional but not pure; its f-vector reads $(14, 40, 32, 2)$. The two three-dimensional facets correspond to the two non-unimodular triangulations of C (arising from splitting every other vertex). The reduced homology is concentrated in dimension two, and we have $H_2(\mathrm{Split}(C_3); \mathbb{Z}) \cong \mathbb{Z}^3$. The graph indicating the compatibility relation among the splits is shown in Figure 2.2.

Figure 2.3 shows three triangulations of C_3. The left one is generated by a totally incompatible system of three splits; that is, it is a facet of $\mathrm{Split}^{\mathrm{w}}(C_3)$ which is not a face of $\mathrm{Split}(C_3)$. The right one is (not unimodular and) generated by a compatible split system (of four vertex splits); that is, it is a facet of both $\mathrm{Split}(C_3)$ and $\mathrm{Split}^{\mathrm{w}}(C_3)$. The middle one is not generated by splits at all.

The triangulation Σ on the left uses only three splits. This examples shows that the converse of Corollary 2.29 is not true, that is, a weakly compatible split system that defines a triangulation does not have to be maximal with respect to cardinality. Furthermore, the triangulation Σ can also be obtained as the common refinement of two non-split coarsest subdivisions. The cell in $\mathrm{SecFan}'(C_3)$ corresponding to Σ is a bipyramid over a triangle. The vertices of this triangle (which is not a face of $\mathrm{SecFan}'(C_3)$) correspond to the three splits, so the relevant cell in $\mathrm{Split}^{\mathrm{w}}(C_3)$ is a triangle, and the apices corresponds to the non-split coarsest subdivisions mentioned above. Since the three splits are totally incompatible there does not exist a corresponding face in $\mathrm{Split}(C_3)$, and the intersection with $\mathrm{Split}(C_3)$ consists of three isolated points.

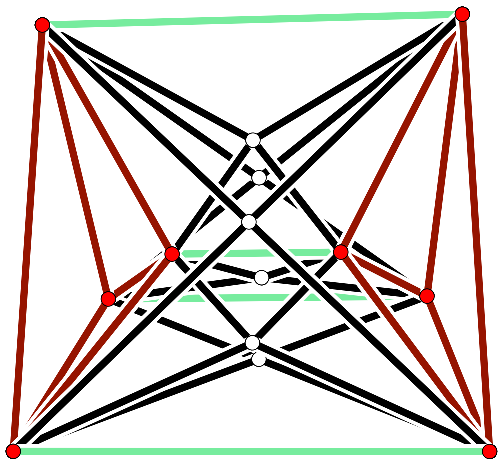

FIGURE 2.2. Compatibility graph of the splits of the regular 3-cube. The four (gray) nodes to the left and the four (gray) nodes to the right correspond to the vertex splits.

A polytopal complex is *zonotopal* if each face is zonotope. A *zonotope* is the Minkowski sum of line segments or, equivalently, the affine projection of a regular cube. Any graph, that is, a one-dimensional polytopal complex, is zonotopal in a trivial way. So especially tight spans of splits and, by Proposition 2.30, of compatible splits systems are zonotopal. In fact, this is even true for arbitrary weakly compatible splits systems. See also Bolker [**14**, Theorem 6.11] and Hirai [**50**, Corollary 2.8].

THEOREM 2.35. *Let $\mathcal{S}$ be a weakly compatible split system on P. Then the tight span $\mathcal{T}_{\mathcal{S}}(P)$ is a (not necessarily pure) zonotopal complex.*

PROOF. Let F be a face of $\mathcal{T}_{\mathcal{S}}(P)$. Since by Lemma 2.23 we have $\mathcal{E}_{\mathcal{S}}(P) = \sum_{S\in\mathcal{S}} \mathcal{E}_{w_S}(P)$, we get (by the same arguments used in the proof of Lemma 2.2) that $F = \sum_{S\in\mathcal{S}} F_S$ for faces F_S of $\mathcal{T}_{w_S}(P)$. The claim now follows from the fact that $\mathcal{T}_{w_S}(P)$ is a line segment for all $S \in \mathcal{S}$. □

A triangulation of a d-polytope is *foldable* if its vertices can be colored with d colors such that each edge of the triangulation receives two distinct colors. This is equivalent to requiring that the dual graph of the triangulation is bipartite; see [**58**, Corollary 11]. Note that foldable simplicial complexes are called "balanced" in [**58**]. The three triangulations of the regular 3-cube in Figure 2.3 are foldable.

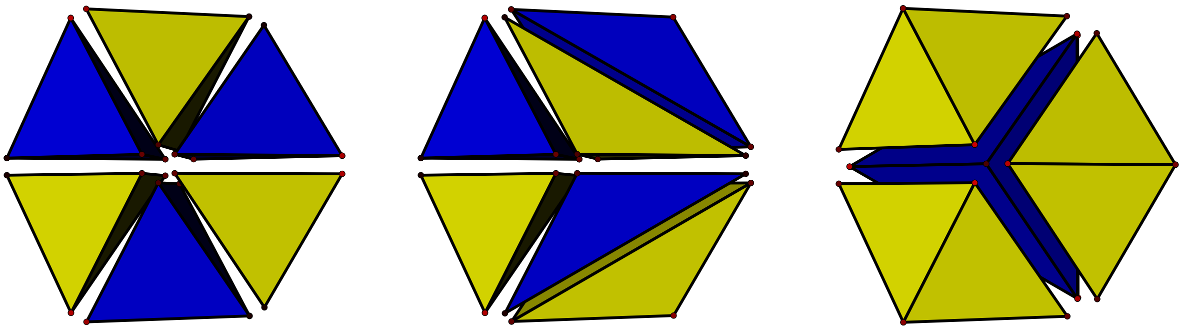

FIGURE 2.3. Three foldable triangulations of the regular 3-cube.

COROLLARY 2.36. *Each split triangulation is foldable.*

PROOF. Let $\mathcal{S}$ be a weakly compatible split system such that $\Sigma_{\mathcal{S}}(P)$ is a triangulation. By Theorem 2.35, each two-dimensional face of the tight span $\mathcal{T}_{\mathcal{S}}(P)$ has an even number of vertices. This implies that $\Sigma_{\mathcal{S}}(P)$ is a triangulation of P such that each of its interior codimension-two-cell is contained in an even number of maximal cells. Now the claim follows from [**58**, Corollary 11]. □

EXAMPLE 2.37. Let C_4 be the four-dimensional cube. In Figure 2.4 there is a picture of the tight span $\mathcal{T}_{\mathcal{S}}(C_4)$ of a split system $\mathcal{S}$ of C_4 with ten weakly compatible splits. As proposed by Theorem 2.35 the complex is zonotopal. It is three-dimensional and its f-vector reads $(24, 36, 14, 1)$. The number of vertices equals $24 = 4!$ which is the normalized volume of C_4, and hence $\Sigma_{\mathcal{S}}(C_4)$ is, in fact, a triangulation. By Corollary 2.36, this triangulation is foldable.

2.5. Hypersimplices

We abbreviate $[n] := \{1, 2, \ldots, n\}$ and $\binom{[n]}{k} := \{X \subseteq [n] \mid |X| = k\}$, as a notational shorthand. The k-th *hypersimplex* in $\mathbb{R}^n$ is defined as

$$\Delta(k, n) \ := \ \left\{ x \in [0,1]^n \ \middle| \ \sum_{i=1}^{n} x_i = k \right\} \ = \ \operatorname{conv} \left\{ \sum_{i\in A} e_i \ \middle| \ A \in \binom{[n]}{k} \right\}.$$

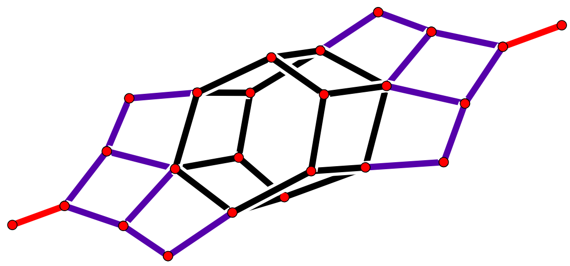

FIGURE 2.4. The tight span of a split triangulation of the 4-cube.

It is $(n-1)$-dimensional and satisfies the conditions of Section 2.2. Throughout the following, we assume that $n \geq 2$ and $1 \leq k \leq n-1$.

A hypersimplex $\Delta(1,n)$ is an $(n-1)$-dimensional simplex. For arbitrary $k \geq 1$ we have $\Delta(k,n) \cong \Delta(n-k,n)$. Moreover, for $p \in [n]$ the equation $x_p = 0$ defines a facet isomorphic to $\Delta(k,n-1)$. And, if $k \geq 2$, the equation $x_p = 1$ defines a facet isomorphic to $\Delta(k-1,n-1)$. This list of facets (induced by the facets of $[0,1]^n$) is exhaustive. Since the hypersimplices are not full-dimensional, the facet defining (affine) hyperplanes are not unique. For the following it will be convenient to work with linear hyperplanes. This way $x_p = 1$ gets replaced by

$$(k-1)x_p \;=\; \sum_{i \in [n] \setminus \{p\}} x_i \,. \tag{2.8}$$

The triplet $(A,B;\mu)$ with $\emptyset \neq A, B \subsetneq [n]$, $A \cup B = [n]$, $A \cap B = \emptyset$, and $\mu \in \mathbb{N}$ defines the linear equation

$$\mu \sum_{i \in A} x_i \;=\; (k-\mu) \sum_{i \in B} x_i \,. \tag{2.9}$$

The corresponding (linear) hyperplane in $\mathbb{R}^n$ is called the $(A,B;\mu)$-*hyperplane*. Clearly, $(A,B;\mu)$ and $(B,A;k-\mu)$ define the same hyperplane. Equation (2.8) corresponds to the $(\{p\},[n] \setminus \{p\};k-1)$-hyperplane.

LEMMA 2.38. *The* $(A,B;\mu)$-hyperplane *is a split hyperplane of* $\Delta(k,n)$ *if and only if* $k-\mu+1 \leq |A| \leq n-\mu-1$ *and* $1 \leq \mu \leq k-1$.

PROOF. It is clear that the $(A,B;\mu)$-hyperplane does not meet the interior of $\Delta(k,n)$ if $\mu \leq 0$ or if $\mu \geq k$. Especially, we may assume that $k \geq 2$.

Suppose now that $|A| \leq k-\mu$. Then each point $x \in \Delta(k,n)$ satisfies $\sum_{i \in A} x_i \leq k-\mu$ and $\sum_{i \in B} x_i \geq k-(k-\mu) = \mu$. This implies that $\mu \sum_{i \in A} x_i \leq (k-\mu) \sum_{i \in B} x_i$, which says that all points in $\Delta(k,n)$ are contained in one of the two halfspaces defined by the $(A,B;\mu)$-hyperplane. Hence it does not define a split. A similar argument shows that $|A| \leq n-\mu-1$ is necessary in order to define a split.

Conversely, assume that $k-\mu+1 \leq |A| \leq n-\mu-1$ and $1 \leq \mu \leq k-1$. We define a point $x \in \Delta(k,n)$ by setting

$$x_i \;:=\; \begin{cases} \frac{k-\mu}{|A|}, & \text{if } i \in A\,, \\ \frac{\mu}{|B|}, & \text{if } i \in B\,. \end{cases}$$

Since $0 < \frac{k-\mu}{|A|} < 1$ and $0 < \frac{\mu}{|B|} < 1$ the point x is contained in the (relative) interior of $\Delta(k,n)$. Moreover, x satisfies Equation (2.9), and so the $(A,B;\mu)$-hyperplane passes through the interior of $\Delta(k,n)$.

It remains to show that the $(A,B;\mu)$-hyperplane does not separate any edge. Let v and w be two adjacent vertices. So we have some $\{p,q\} \in \binom{[n]}{2}$ with $v - w = e_p - e_q$. Aiming at an indirect argument, we assume that v and w are on opposite sides of the $(A,B;\mu)$-hyperplane, that is, without loss of generality $\mu\sum_{i\in A} v_i > (k-\mu)\sum_{i\in B} v_i$ and $\mu\sum_{i\in A} w_i < (k-\mu)\sum_{i\in B} w_i$. This gives

$$0 \;<\; \mu\sum_{i\in A} v_i - (k-\mu)\sum_{i\in B} v_i \;=\; \mu(\chi_A(p) - \chi_A(q))$$

and

$$0 \;<\; (k-\mu)\sum_{i\in B} w_i - \mu\sum_{i\in A} w_i \;=\; (k-\mu)(\chi_B(p) - \chi_B(q)),$$

where characteristic functions are denoted as $\chi_\cdot(\cdot)$. Since $\mu > 0$ and $\mu < k$ it follows that $\chi_A(q) < \chi_A(p)$ and $\chi_B(q) < \chi_B(p)$. Now the characteristic functions take values in $\{0,1\}$ only, and we arrive at $\chi_A(q) = \chi_B(q) = 0$ and $\chi_A(p) = \chi_B(p) = 1$. Both these equations contradict the fact that (A,B) is a partition of $[n]$. So we conclude that, indeed, the $(A,B;\mu)$-hyperplane defines a split. □

This allows to characterize the splits of the hypersimplices.

PROPOSITION 2.39. *Each split hyperplane of* $\Delta(k,n)$ *is defined by a linear equation of the type* (2.9).

PROOF. Using Observation 2.12 and exploiting the fact that facets of hypersimplices are hypersimplices, we can proceed by induction on n and k as follows.

Our induction is based on the case $k = 1$. Since $\Delta(1,n)$ is an $(n-1)$-simplex, which does not have any splits, the claim is trivially satisfied. The same holds for $k = n-1$ as $\Delta(n-1,n) \cong \Delta(1,n)$.

For the rest of the proof we assume that $2 \le k \le n-2$. In particular, this implies that $n \ge 4$.

Let $\sum_{i\in[n]} \alpha_i x_i = 0$ define a split hyperplane H of $\Delta(k,n)$. The facet defining hyperplane $F_p = \{x \mid x_p = 0\}$ is intersected by H, and we have

$$F_p \cap H \;=\; \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i\in[n]\setminus\{p\}} \alpha_i x_i = 0 = x_p \right\}.$$

Three cases arise:

(a) $F_p \cap H$ is a facet of $F_p \cap \Delta(k,n) \cong \Delta(k,n-1)$ defined by $x_q = 0$ (with $q \ne p$),
(b) $F_p \cap H$ is a facet of $F_p \cap \Delta(k,n) \cong \Delta(k,n-1)$ as defined by Equation (2.8), or
(c) $F_p \cap H$ defines a split of $F_p \cap \Delta(k,n) \cong \Delta(k,n-1)$.

If $F_p \cap H$ is of type (a), then $\alpha_i = 0$ for all $i \ne p$ and $\alpha_p \ne 0$. As not all the α_i can vanish there is at most one $p \in [n]$ such that $F_p \cap H$ is of type (a). Since assumed that $n \ge 4$ there are at least two distinct $p,q \in [n]$ such that $F_p \cap H$ and $F_q \cap H$ are of type (b) or (c). By symmetry, we can further assume that $p = 1$ and $q = n$. So we get a partition

(A, B) of $[n-1]$ and a partition (A', B') of $\{2, 3, \ldots, n\}$ with $\mu, \mu' \in \mathbb{N}$ such that $F_1 \cap H$ is defined by $x_1 = 0$ and

$$\mu \sum_{i \in A} x_i \;=\; (k-\mu) \sum_{i \in B} x_i \,,$$

while $F_n \cap H$ is defined by $x_n = 0$ and

$$\mu' \sum_{i \in A'} x_i \;=\; (k-\mu') \sum_{i \in B'} x_i \,.$$

We infer that there is a real number λ such that $\alpha_i = \lambda\mu$ for all $i \in A$, $\alpha_i = \lambda(k-\mu)$ for all $i \in B$. It remains to show that $\alpha_n \in \{\lambda\mu, \lambda(k-\mu)\}$. Similarly, there is a real number λ' such that $\alpha_i = \lambda'\mu'$ for all $i \in A'$, $\alpha_i = \lambda'(k-\mu')$ for all $i \in B'$. As $n \geq 4$ we have $A \cap A' \neq \emptyset$ or $B \cap B' \neq \emptyset$. We obtain $\alpha_i = \lambda\mu = \lambda'\mu'$ for $i \in (A \cap A') \cup (B \cap B')$. Finally, this shows that $\alpha_n \in \{\lambda'\mu', \lambda'(k-\mu')\} = \{\lambda\mu, \lambda(k-\mu)\}$. This completes the proof. □

Theorem 2.40. *The total number of splits of* $\Delta(k, n)$ *(with* $k \leq n/2$*) equals*

$$(k-1)\left(2^{n-1} - (n+1)\right) - \sum_{i=2}^{k-1} (k-i)\binom{n}{i} .$$

Proof. We have to count the $(A, B; \mu)$-hyperplanes with the restrictions listed in Lemma 2.38. So we take a set $A \subset [n]$ with at least two and at most $n-2$ elements. If A has cardinality i, then there are $\min(k-1, n-i-1) - \max(1, k-i+1) + 1$ choices for μ. Recall that $(A, B; \mu)$ and $(B, A; k-\mu)$ define the same split; in this way we have counted each split twice. So we get

$$\frac{1}{2} \sum_{i=2}^{n-2} \left(\min(k, n-i) - \max(1, k-i+1) \right)\binom{n}{i} \;=\; \frac{1}{2} \sum_{i=2}^{n-2} \left(\min(i, k, n-i) - 1 \right)\binom{n}{i}$$

splits, where the equality holds since $k \leq n/2$. For a further simplification we rewrite the sum to get

$$\begin{aligned}
&\frac{1}{2} \sum_{i=2}^{k-1} (i-1)\binom{n}{i} + \frac{1}{2} \sum_{i=k}^{n-k} (k-1)\binom{n}{i} + \frac{1}{2} \sum_{i=n-k+1}^{n-2} (n-i-1)\binom{n}{i} \\
&\quad= \frac{1}{2}(k-1) \sum_{i=2}^{n-2} \binom{n}{i} + \frac{1}{2} \sum_{i=2}^{k-1} (i-1-(k-1))\binom{n}{i} + \frac{1}{2} \sum_{i=n-k+1}^{n-2} (n-i-1-(k-1))\binom{n}{i} \\
&\quad= (k-1)\left(2^{n-1} - (n+1)\right) - \sum_{i=2}^{k-1} (k-i)\binom{n}{i} .
\end{aligned}$$

□

If we have two distinct splits $(A, B; \mu)$ and $(C, D; \nu)$ then either $\{A \cap C, A \cap D, B \cap C, B \cap D\}$ is a partition of $[n]$ into four parts, or exactly one of the four intersections is empty. If, for instance, $B \cap D = \emptyset$ then $B \subseteq C$ and $D \subseteq A$.

Proposition 2.41. *Two splits* $(A, B; \mu)$ *and* $(C, D; \nu)$ *of* $\Delta(k, n)$ *are compatible if and only if one of the following holds:*

$$\begin{array}{rcl} |A \cap C| &\leq& k - \mu - \nu\,, \\ |B \cap C| &\leq& \mu - \nu\,, \quad \textit{or} \end{array} \qquad \begin{array}{rcl} |A \cap D| &\leq& \nu - \mu\,, \\ |B \cap D| &\leq& \mu + \nu - k\,. \end{array}$$

For an arbitrary set $I \subseteq [n]$ we abbreviate $x_I := \sum_{i \in I} x_i$. In particular, $x_\emptyset = 0$ and for $x \in \Delta(k, n)$ one has $x_{[n]} = k$.

Proof. Let $x \in \Delta(k, n)$ be in the intersection of the $(A, B; \mu)$-hyperplane and the $(C, D; \nu)$-hyperplane. Our split equations take the form

$$\begin{array}{rcl} \mu(x_{A\cap C} + x_{A\cap D}) &=& (k - \mu)(x_{B\cap C} + x_{B\cap D}) \quad \text{and} \\ \nu(x_{A\cap C} + x_{B\cap C}) &=& (k - \nu)(x_{A\cap D} + x_{B\cap D})\,. \end{array}$$

In view of $(A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D) = [n]$ we additionally have $x_{A\cap C} + x_{A\cap D} + x_{B\cap C} + x_{B\cap D} = k$, and thus we arrive at the equivalent system of linear equations

$$x_{A\cap C} = k - \mu - \nu + x_{B\cap D}\,, \quad x_{A\cap D} = \nu - x_{B\cap D}\,, \quad \text{and} \quad x_{B\cap C} = \mu - x_{B\cap D}\,. \tag{2.10}$$

Now the two given splits are *incompatible* if and only if there exists a point $x \in (0, 1)^n$ satisfying the conditions (2.10).

Suppose first that none of the four intersections $A \cap C$, $A \cap D$, $B \cap C$, and $B \cap D$ is empty. Then $x \in (0, 1)^n$ satisfies the Equations (2.10) if and only if the system of inequalities in $x_{B\cap D}$

$$\begin{array}{rclcl} 0 &<& x_{B\cap D} &<& |B \cap D| \\ 0 &<& k - \mu - \nu + x_{B\cap D} &<& |A \cap C| \\ 0 &<& \mu - x_{B\cap D} &<& |B \cap C| \\ 0 &<& \nu - x_{B\cap D} &<& |A \cap D| \end{array} \tag{2.11}$$

has a solution. This is equivalent to the following system of inequalities:

$$\begin{array}{lclcl} 0 &<& x_{B\cap D} &<& |B \cap D| \\ \mu + \nu - k &<& x_{B\cap D} &<& |A \cap C| + \mu + \nu - k \\ \mu - |B \cap C| &<& x_{B\cap D} &<& \mu \\ \nu - |A \cap D| &<& x_{B\cap D} &<& \nu\,. \end{array}$$

Obviously, the latter system admits a solution if and only if each of the four terms on the left is smaller than each of the four terms on the right. Most of the resulting 16 inequalities are redundant. The following four inequalities remain

$$\begin{array}{rcl} |A \cap C| &>& k - \mu - \nu \\ |A \cap D| &>& \nu - \mu \\ |B \cap C| &>& \mu - \nu \\ |B \cap D| &>& \mu + \nu - k\,, \end{array}$$

and this completes the proof of this case.

For the remaining cases, we can assume by symmetry that $A \cap C = \emptyset$. Then $x \in (0, 1)^n$ satisfies the Equations (2.10) if and only if $x_{B\cap D} = \mu + \nu - k$, $x_{A\cap D} = k - \mu$, and $x_{B\cap C} = k - \nu$.

So the splits are not compatible if and only if

$$\begin{array}{lllll} 0 & < & k-\mu & < & |A \cap D| = |A| \\ 0 & < & k-\nu & < & |B \cap C| = |C| \\ 0 & < & \mu+\nu-k & < & |B \cap D| \, . \end{array}$$

Since, by Lemma 2.38, the first two inequalities hold for all splits this proves that the splits are compatible if and only if

$$|A \cap C| = 0 \le k - \mu - \nu \quad \text{or} \quad |B \cap D| \le \mu + \nu - k.$$

However, again by using Lemma 2.38, one has $|A \cap D| = |A| > k - \mu > \nu - \mu$, which implies $|A \cap D| \le \nu - \mu$ and, similarly, $|B \cap C| \le \mu - \nu$ cannot be true. This completes the proof. □

In fact, the four cases of the proposition are equivalent in the sense that, by renaming the four sets and exchanging μ and ν or μ and $k-\mu$ in a suitable way, one will always be in the first case.

Example 2.42. We consider the case $k = 3$ and $n = 6$. For instance, the splits $(\{1,2,6\},\{3,4,5\};2)$ and $(\{4,5,6\},\{1,2,3\};2)$ are compatible since the intersection $\{3,4,5\} \cap \{1,2,3\} = \{3\}$ has only one element and $2+2-3 = 1$, that is, the inequality "$|C \cap D| \le \mu + \nu - k$" is satisfied.

Corollary 2.43. *Two splits $(A, B; \mu)$ and $(A, B; \nu)$ of $\Delta(k, n)$ are always compatible.*

Proof. Without loss of generality we can assume that $\mu \ge \nu$. Then the condition "$|B \cap C| \le \mu - \nu$" of Proposition 2.41 is satisfied. □

In Proposition 2.56 below, we will show that the 1-skeleton of the weak split complex of any hypersimplex is always a complete graph. In particular, the weak split complex of $\Delta(k, n)$ is connected. (Or it is void if $k \in \{1, n-1\}$.)

2.6. Finite Metric Spaces

This section revisits the classical case, studied by Bandelt and Dress [**3, 4**]; see also Isbell [**55**]. Its purpose is to show how some of the key results can be obtained as immediate corollaries to our results above.

Let $d : \binom{[n]}{2} \to \mathbb{R}_{\ge 0}$ be a metric on the finite set $[n]$; that is, d is a symmetric dissimilarity function which obeys the triangle inequality. By setting

$$w_d(e_i + e_j) \;:=\; -d(i, j)\,,$$

each metric d defines a weight function w_d on the second hypersimplex $\Delta(2, n)$. Hence the results for $k = 2$ from Section 2.5 can be applied here. The *tight span* of d is the tight span $\mathcal{T}_{w_d}(\Delta(2, n))$.

Let $S = (A, B)$ be a *split partition* of the set $[n]$, that is, $A, B \subseteq [n]$ with $A \cup B = [n]$, $A \cap B = \emptyset$, $|A| \ge 2$, and $|B| \ge 2$. This gives rise to the *split metric*

$$d_S(i, j) \;:=\; \begin{cases} 0 & \text{if } \{i, j\} \subseteq A \text{ or } \{i, j\} \subseteq B, \\ 1 & \text{otherwise.} \end{cases}$$

The weight function $w_{d_S} = -d_S$ induces a split of the second hypersimplex $\Delta(2, n)$, which is induced by the $(A, B; 1)$-hyperplane defined in Equation (2.9). Proposition 2.39 now implies the following characterization.

COROLLARY 2.44. *Each split of* $\Delta(2, n)$ *is induced by a split metric.*

Specializing the formula in Theorem 2.40 with $k = 2$ gives the following.

COROLLARY 2.45. *The hypersimplex* $\Delta(2, n)$ *has* $2^{n-1} - n - 1$ *splits.*

The following corollary and proposition show that our notions of compatibility and weak compatibility agree with those of Bandelt and Dress [**4**] for in the special case of $\Delta(2, n)$.

COROLLARY 2.46 (Hirai [**50**], Proposition 4.16). *Two splits* (A, B) *and* (C, D) *of* $\Delta(2, n)$ *are compatible if and only if one of the four sets* $A \cap C$, $A \cap D$, $B \cap C$, *and* $B \cap D$ *is empty.*

PROOF. Let (A, B) and (C, D) be splits of $\Delta(2, n)$. We are in the situation of Proposition 2.41 with $k = 2$ and $\mu = \nu = 1$. Hence all the right hand sides of the four inequalities in Proposition 2.41 yield zero, and this gives the claim. □

For a split $S = (A, B)$ of $\Delta(2, n)$ and $m \in [n]$ we denote by $S(m)$ that of the two sets A, B with $m \in S(m)$.

PROPOSITION 2.47. *A set* $\mathcal{S}$ *of splits of* $\Delta(2, n)$ *is weakly compatible if and only if there does not exist* $m_0, m_1, m_2, m_3 \in [n]$ *and* $S_1, S_2, S_3 \in \mathcal{S}$ *such that* $S_i(m_0) = S_j(m_i)$ *if and only if* $i = j$.

PROOF. This is the definition of a weakly compatible split system $\Delta(2, n)$ originally given by Bandelt and Dress in [**4**, Section 1, page 52]. Their Corollary 10 states that $\mathcal{S}$ is weakly compatible in their sense if and only if $\sum_{S \in \mathcal{S}} w_S$ is a coherent decomposition. However, this is our definition of weakly compatibility according to Lemma 2.23. □

EXAMPLE 2.48. The hypersimplex $\Delta(2, 4)$ is the regular octahedron, already studied in Example 2.33. It has the three splits $(\{1, 2\}, \{3, 4\})$, $(\{1, 3\}, \{2, 4\})$, and $(\{1, 4\}, \{2, 3\})$. The weak split complex is a triangle, and the split compatibility graph consists of three isolated points.

The split compatibility graph of $\Delta(2, 5)$ is isomorphic to the Petersen graph, which is shown in Figure 2.5.

By Proposition 2.30, each compatible system of splits gives rise to a tree. On the other hand, given a tree with n labeled leaves take for each edge E that is not connected to a leave the split (A, B) where A is the set of labels on one side of E and B the set of labels on the other side. So each tree gives rise to a system of splits for $\Delta(2, k)$ which is easily seen to be compatible. This argument can be augmented to a proof of the following theorem.

THEOREM 2.49 (Buneman [**15**]; Billera, Holmes, and Vogtmann [**8**]). *The split complex* $\mathrm{Split}(\Delta(2, n))$ *is the complex of trivalent leaf-labeled trees with* n *leaves.*

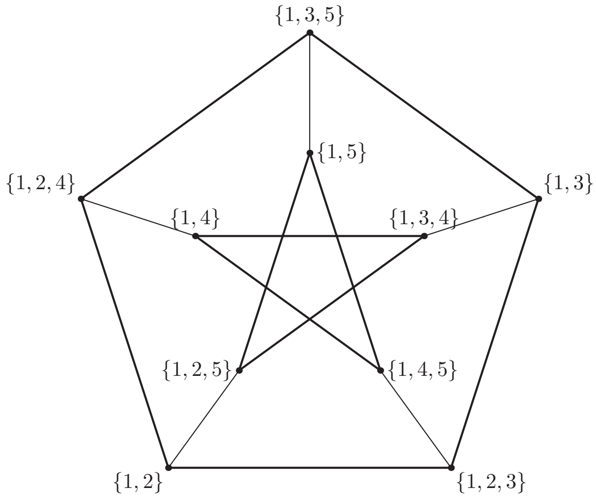

FIGURE 2.5. Split compatibility graph of $\Delta(2,5)$; a split (A,B) with $1 \in A$ is labeled "A".

The split complex $\mathrm{Split}(\Delta(2,n))$ is equal to the *link of the origin* L_{n-1} of the *space of phylogenetic trees* in [**8**]. It was proved in [**93**, Theorem 2.4] (see also Robinson and Whitehouse [**79**]) that $\mathrm{Split}(\Delta(2,n))$ is homotopy equivalent to a wedge of $n-3$ spheres. By a result of Trappmann and Ziegler, $\mathrm{Split}(\Delta(2,n))$ is even shellable [**92**]. Markwig and Yu [**69**] recently identified the space of k tropically collinear points in the tropical $(d-1)$-dimensional affine space as a (shellable) subcomplex of $\mathrm{Split}(\Delta(2,k+d))$.

EXAMPLE 2.50. Let $\mathcal{S} = \{(A_{ij}, [n] \setminus A_{ij}) \mid 1 \le i < j \le n \text{ and } j-i < n-2\}$ where $A_{ij} := \{i, i+1, \ldots, j-1, j\}$ be a split system for the hypersimplex $\Delta(2,n)$. The combinatorial criterion of Proposition 2.47 shows that this split system is weakly compatible, and that $|\mathcal{S}| = \binom{n}{2} - n$. Since $\Delta(2,n)$ has $\binom{n}{2}$ vertices and is of dimension $n-1$, Corollary 2.29 implies that $\Sigma_{\mathcal{S}}(P)$ is a triangulation. This triangulation is known as the *thrackle triangulation* in the literature; see [**20**], [**89**, Chapter 14], and additionally [**86, 4, 67, 46**] for further occurrences of this triangulation. In fact, as one can conclude from [**26**, Theorem 3.1] in connection with [**4**, Theorem 5], this is the only split triangulation of $\Delta(2,n)$, up to symmetry.

2.7. Matroid Polytopes and Tropical Grassmannians

In the following, we copy some information from Speyer and Sturmfels [**84**]; the reader is referred to this source for the details.

Let $\mathbb{Z}[p] := \mathbb{Z}[p_{i_1,\ldots,i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n]$ be the polynomial ring in $\binom{n}{k}$ indeterminates with integer coefficients. The indeterminate $p_{i_1,\ldots,i_k}$ can be identified with the $k \times k$-minor of a $k \times n$-matrix with columns numbered $(i_1, i_2, \ldots, i_k)$. The *Plücker ideal* $I_{k,n}$ is defined as the ideal generated by the algebraic relations among these minors. It is obviously homogeneous and is known to be a prime ideal. For an algebraically closed field $\mathbb{K}$ the projective variety defined by $I_{k,n} \otimes_{\mathbb{Z}} \mathbb{K}$ in the polynomial ring $\mathbb{K}[p] = \mathbb{Z}[p] \otimes_{\mathbb{Z}} \mathbb{K}$ is the *Grassmannian* $G_{k,n}$ (over $\mathbb{K}$). It parametrizes the k-dimensional linear subspaces of the vector space $\mathbb{K}^n$.

For instance, we can pick $\mathbb{K}$ as the algebraic closure of the field $\mathbb{C}(t)$ of rational functions. Then for an arbitrary ideal I in $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_m]$ its *tropicalization* $\mathcal{T}(I)$ is the set of all vectors $w \in \mathbb{R}^m$ such that the initial ideal $\mathrm{in}_w(I)$ with respect to the term order defined by the weight function w does not contain any monomial. The *tropical Grassmannian* $\mathrm{Gr}(k, n)$ (over $\mathbb{K}$) is the tropicalization of the Plücker ideal $I_{k,n} \otimes_{\mathbb{Z}} \mathbb{K}$.

The tropical Grassmannian $\mathrm{Gr}(k, n)$ is a polyhedral fan in $\mathbb{R}^{\binom{n}{k}}$ such that each of its maximal cones has dimension $(n-k)k+1$. In a way the fan $\mathrm{Gr}(k, n)$ contains redundant information. We describe the three step reduction in [**84**, Section 3].

Let φ be the linear map from $\mathbb{R}^n$ to $\mathbb{R}^{\binom{n}{k}}$ which sends $x = (x_1, \ldots, x_n)$ to $(x_I \mid I \in \binom{[n]}{k})$. Recall that x_I is defined as $\sum_{i \in I} x_i$. The map φ is injective, and its image $\mathrm{im}\,\varphi$ coincides with the intersection of all maximal cones in $\mathrm{Gr}(k, n)$. Moreover, the vector $\mathbb{1} := (1, 1, \ldots, 1)$ of length $\binom{n}{k}$ is contained in the image of φ. This leads to the definition of the two quotient fans

$$\mathrm{Gr}'(k, n) \ := \ \mathrm{Gr}(k, n)/\mathbb{R}\mathbb{1} \quad \text{and} \quad \mathrm{Gr}''(k, n) \ := \ \mathrm{Gr}(k, n)/\,\mathrm{im}\,\varphi\,.$$

Finally, let $\mathrm{Gr}'''(k, n)$ be the (spherical) polytopal complex arising from intersecting $\mathrm{Gr}''(k, n)$ with the unit sphere in $\mathbb{R}^{\binom{n}{k}}/\,\mathrm{im}\,\varphi$. We have $\dim \mathrm{Gr}'''(k, n) = n(k-1) - k^2$. It seems to be common practice to use the name "tropical Grassmannian" interchangeably for $\mathrm{Gr}(k, n)$, $\mathrm{Gr}'(k, n)$, $\mathrm{Gr}''(k, n)$, as well as $\mathrm{Gr}'''(k, n)$.

It is unlikely that it is possible to give a complete combinatorial description of all tropical Grassmannians. The contribution of combinatorics here is to provide kind of an "approximation" to the tropical Grassmannians via matroid theory. For a background on matroids, see the books edited by White [**94, 95**].

The *Dressian* $\mathrm{Dr}(k, n)$ is the subfan of the secondary fan of $\Delta(k, n)$ of those weight functions which induce matroid subdivisions. A polytopal subdivision Σ of $\Delta(k, n)$ is a *matroid subdivision* if each (maximal) cell is a matroid polytope. If M is a matroid on the set $[n]$ then the corresponding *matroid polytope* is the convex hull of those $0/1$-vectors in $\mathbb{R}^n$ which are characteristic functions of the bases of M. A finite point set $X \subset \mathbb{R}^d$ (possibly with multiple points) gives rise to a matroid $\mathcal{M}(X)$ by taking as bases for $\mathcal{M}(X)$ the maximal affinely independent subsets of X. The following characterization of matroid subdivisions is essential.

THEOREM 2.51 (Gel′fand, Goresky, MacPherson, and Serganova [**38**], Theorem 4.1). *Let Σ be a polytopal subdivision of $\Delta(k, n)$. The following are equivalent:*

(a) *The maximal cells of Σ are matroid polytopes, that is, Σ is a matroid subdivision,*
(b) *the 1-skeleton of Σ coincides with the 1-skeleton of $\Delta(k, n)$, and*
(c) *the edges in Σ are parallel to the edges of $\Delta(k, n)$.*

Regular matroid subdivisions of hypersimplices are called "generalized Lie complexes" by Kapranov [**60**]. The corresponding equivalence classes of weight functions are the "tropical Plücker vectors" of Speyer [**83**].

The relationship between the two fans $\mathrm{Dr}(k, n)$ and $\mathrm{Gr}(k, n)$ is the following. Algebraically, $\mathrm{Dr}(k, n)$ is the tropicalization of the ideal of quadratic Plücker relations; see Speyer [**83**, Section 2]. Conversely, each weight function in the fan $\mathrm{Gr}(k, n)$ gives rise to a matroid subdivision of $\Delta(k,n)$. However, since there is no secondary fan naturally associated with $\mathrm{Gr}(k,n)$ it is a priori not clear how $\mathrm{Gr}(k,n)$ sits inside $\mathrm{Dr}(k,n)$. Note that, unlike $\mathrm{Gr}(k,n)$, the Dressian does not depend on the characteristic of the field $\mathbb{K}$.

Our goal for the rest of this section is to explain how the hypersimplex splits are related to the Dressians.

PROPOSITION 2.52. *Let Σ be a matroid subdivision and S a split of $\Delta(k,n)$. Then Σ and S have a common refinement (without new vertices).*

PROOF. Of course, one can form the common refinement Σ' of Σ and S but Σ' may contain additional vertices, and hence does not have to be a polytopal subdivision of $\Delta(k,n)$. However, additional vertices can only occur if some edge of Σ is cut by the hyperplane H_S. By Theorem 2.51, all edges of Σ are edges of $\Delta(k,n)$. But since S is a split, it does not cut any edges of $\Delta(k,n)$. Therefore Σ' is a common refinement of S and Σ without new vertices. □

In order to continue, we recall some notions from linear algebra: Let V be a vector space. A set $A \subset V$ is said to be in *general position* if any subset S of A with $|S| \leq \dim V + 1$ is affinely independent. A family $\mathcal{A} = \{A_i \mid i \in I\}$ of sets in V is said to be in *relative general position* if for each affinely dependent set $S \subseteq \bigcup_{i\in I} A_i$ with $|S| \leq \dim V + 1$ there exists some $i \in I$ such that $S \cap A_i$ is affinely dependent.

LEMMA 2.53. *Let $\mathcal{M}(X)$ be a matroid of rank k defined by $X \subset \mathbb{R}^{k-1}$. If $X = \bigcup_{i\in I} A_i$ for some family $\mathcal{A} = \{A_i \mid i \in I\}$ of sets in relative general position such that each A_i is in general position as a subset of $\operatorname{aff} A_i$, then the set of bases of $\mathcal{M}(X)$ is given by*

$$\{B \subset X \mid |B| = k \text{ and } |B \cap A_i| \leq \dim(\operatorname{aff} A_i) + 1 \text{ for all } i \in I\}\ . \tag{2.12}$$

PROOF. It is obvious that for each basis B of $\mathcal{M}(X)$ one has $|B \cap A_i| \leq \dim(\operatorname{aff} A_i) + 1$ for all $i \in I$. So it remains to show that each set B in (2.12) is affinely independent. Let B be such a set and suppose that B is not affinely independent. Since $\mathcal{A}$ is in relative general position, there exists some $i \in I$ such that $B \cap A_i$ is affinely dependent. However, since $|B \cap A_i| \leq \dim(\operatorname{aff} A_i) + 1$, this contradicts the fact that A_i is in general position in $\operatorname{aff} A_i$. □

From each split $(A, B; \mu)$ of $\Delta(k,n)$ we construct two matroid polytopes with points labeled by $[n]$: Take any $(\mu - 1)$-dimensional (affine) subspace $U \subset \mathbb{R}^{k-1}$ and put $|B|$ points labeled by B into U such that they are in general position (as a subset of U). The remaining points, labeled by A, are placed in $\mathbb{R}^{k-1} \setminus U$ such that they are in general position and in relative general position together with the set of points labeled by B. By Lemma 2.53, the bases of the corresponding matroid are all k-element subsets of $[n]$ with at most μ points in B. These are exactly the points on one side of the $(A, B; \mu)$-hyperplane. The second matroid is obtained symmetrically, that is, starting with $|A|$ points in a $(k - \mu - 1)$-dimensional subspace. Since splits are regular and correspond to rays in the secondary fan, we have proved the following lemma.

LEMMA 2.54. *Each split of $\Delta(k,n)$ defines a regular matroid subdivision and hence a ray in $\mathrm{Dr}(k,n)$.*

Matroids arising in this way are called *split matroids*, and the corresponding matroid polytopes are the *split matroid polytopes*.

REMARK 2.55. Kim [**62**] studies the splits of general matroid polytopes. However, his definition of a split requires that it induces a matroid subdivision. Lemma 2.54 shows that for the entire hypersimplex these notions agree. In this case, [**62**, Theorem 4.1] reduces to our Lemma 2.38.

PROPOSITION 2.56. *The* 1*-skeleton of the weak split complex* $\mathrm{Split}^{\mathrm{w}}(\Delta(k,n))$ *of* $\Delta(k,n)$ *is a complete graph.*

PROOF. We have to prove that any two splits of $\Delta(k,n)$ are weakly compatible. Since splits are matroid subdivisions by Lemma 2.54, this immediately follows from Proposition 2.52. □

EXAMPLE 2.57. We continue our Example 2.48, where $k = 2$ and $n = 4$. Up to symmetry, each split of the regular octahedron $\Delta(2,4)$ looks like $(\{1,2\},\{3,4\};1)$, that is, so we have $\mu = 1$.

In this case, the affine subspace U is just a single point on the line $\mathbb{R}^1$. The only choice for the two points corresponding to $B = \{3,4\}$ is the point U itself. The two points corresponding to $A = \{1,2\}$ are two arbitrary distinct points both of which are distinct from U. The situation is displayed in Figure 2.6 on the left. This defines the first of the two matroids induced by the split $(\{1,2\},\{3,4\};1)$. Its bases are $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, and $\{2,4\}$.

The second matroid is obtained in a similar way. Both matroid polytopes are square pyramids, and they are shown (with their vertices labeled) in Figure 2.6 on the right. The pyramid in bold is the one corresponding to the matroid whose construction has been explained in detail above and which is shown on the left.

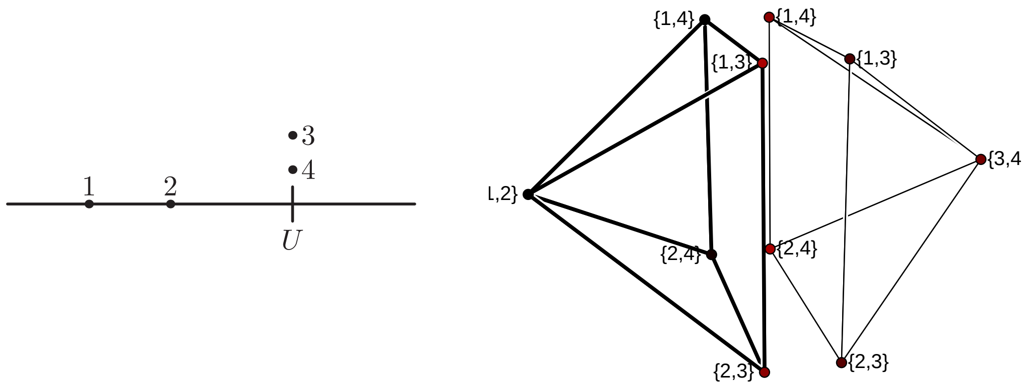

FIGURE 2.6. Matroid and matroid subdivision induced by a split as explained in Example 2.57.

As in the case of the tropical Grassmannian, we can intersect the fan $\mathrm{Dr}(k,n)$ with the unit sphere in $\mathbb{R}^{\binom{n}{k}-n}$ to arrive at a (spherical) polytopal complex $\mathrm{Dr}'(k,n)$, which we also call the Dressian. The following is one of our main results.

THEOREM 2.58. *The split complex* $\mathrm{Split}(\Delta(k,n))$ *is a polytopal subcomplex of the Dressian* $\mathrm{Dr}'(k,n)$.

PROOF. By Proposition 2.26, the split complex is a subcomplex of $\mathrm{SecFan}'(\Delta(k,n))$. Furthermore, by Lemma 2.54 each split corresponds to a ray of $\mathrm{Dr}(k,n)$. So it remains to show that all maximal cells of $\Sigma_{\mathcal{S}}(\Delta(k,n))$ are matroid polytopes whenever $\mathcal{S}$ is a compatible system of splits. The proof will proceed by induction on k and n. Note that, since $\Delta(k,n) \cong \Delta(n-k,n)$, it is enough to have as base case $k = 2$ and arbitrary n, which is given by Proposition 2.61.

By Theorem 2.51, we have to show that there do not occur any edges in $\Sigma_{\mathcal{S}}(\Delta(k,n))$ that are not edges of $\Delta(k,n)$. Since $\mathcal{S}$ is compatible no split hyperplanes meet in the interior of $\Delta(k,n)$, and so additional edges could only occur in the boundary. By Observation 2.12, for each split $S \in \mathcal{S}$ and each facet F of $\Delta(k,n)$ there are two possibilities: Either H_S does not meet the interior of F, or H_S induces a split S' on F. The restriction of $\Sigma_{\mathcal{S}}(\Delta(k,n))$ to F equals the common refinement of all such splits S'. So, using the induction hypothesis and again Theorem 2.51, it suffices to prove that the split systems that arise in this fashion are compatible.

So let $S = (A, B; \mu) \in \mathcal{S}$. We have to consider two types of facets of $\Delta(k,n)$ induced by $x_i = 0$, $x_i = 1$, respectively. In the first case, the arising facet F is isomorphic to $\Delta(k, n-1)$ and, if H_S meets F in the interior, the split S' of F equals $(A \setminus \{i\}, B; \mu)$ or $(A, B \setminus \{i\}; \mu)$. It is now obvious by Proposition 2.41 that the system of all such S' is compatible if $\mathcal{S}$ was.

In the second case, the facet F is isomorphic to $\Delta(k-1, n-1)$ and S' (again if H_S meets the interior of F at all) equals $(A \setminus \{i\}, B; \mu)$ or $(A, B \setminus \{i\}; \mu - 1)$. To show that a split system is compatible, it suffices to show that any two of its splits are compatible. So let $S = (A, B; \mu)$ and $T = (C, D; \nu)$ be compatible splits for $\Delta(k,n)$ such that H_S and H_T meet the interior of F, and $S' = (A', B'; \mu')$, $T' = (C', D'; \nu')$, respectively, the corresponding splits of F. By the remark after Proposition 2.41, we can suppose that we are in the first case of Proposition 2.41, that is, $|A \cap C| \leq k - \mu - \nu$. We now have to consider the four cases that i is an element of either $A \cap C$, $A \cap D$, $B \cap C$, or $B \cap D$. In the first case, we have $S' = (A \setminus \{i\}, B; \mu)$ and $T' = (C \setminus \{i\}, D, \nu)$. We get $|A' \cap C'| = |A \cap C| - 1 \leq k - \mu - \nu - 1 = k - 1 - \mu' - \nu'$, so S' and T' are compatible. The other cases follow similarly, and this completes the proof of the theorem. □

CONSTRUCTION 2.59. We will now explicitly construct the matroid polytopes that occur in the refinement of two compatible splits. So consider two compatible splits of $\Delta(k,n)$ defined by an $(A, B; \mu)$- and a $(C, D; \nu)$-hyperplane. These two hyperplanes divide the space into four (closed) regions. Compatibility implies that the intersection of one of these regions with $\Delta(k,n)$ is not full-dimensional, two of the intersections are split matroid polytopes, and the last one is a full-dimensional polytope of which we have to show that it is a matroid polytope. It therefore suffices to show that one of the four intersections is a full-dimensional matroid polytope that is not a split matroid polytope.

By Proposition 2.41 and the remark following its proof, we can assume without loss of generality that $|B \cap D| \leq \mu + \nu - k$. Note first that the equation $\sum_{i \in B} x_i = \mu$ also defines the $(A, B; \mu)$-hyperplane from Equation (2.9) since $x_{A \cup B} = k$ for any point $x \in \Delta(k,n)$.

We will show that the intersection of $\Delta(k,n)$ with the two halfspaces defined by

$$\sum_{i\in B} x_i \;\le\; \mu \qquad \text{and} \qquad \sum_{i\in D} x_i \;\le\; \nu$$

is a full-dimensional matroid polytope which is not a split matroid polytope.

To this end, we define a matroid on the ground set $[n]$ together with a realization in $\mathbb{R}^{k-1}$ as follows. Pick a pair of (affine) subspaces U_B and U_D of $\mathbb{R}^{k-1}$ such that the following holds: $\dim U_B = \mu - 1$, $\dim U_D = \nu - 1$, and $\dim(U_B \cap U_D) = \mu + \nu - k - 1$. Note that the last expression is non-negative as $0 \le |B \cap D| \le \mu + \nu - k - 1$. The dimension formula then implies that $\dim(U_B + U_D) = \mu - 1 + \nu - 1 - \mu - \nu + k + 1 = k - 1$, that is, $U_B + U_D = \mathbb{R}^{k-1}$.

Each element in $[n]$ labels a point in $\mathbb{R}^{k-1}$ according to the following restrictions. For each element in the intersection $B \cap D$ we pick a point in $U_B \cap U_D$ such that the points with labels in $B \cap D$ are in general position within $U_B \cap U_D$. Since $|B \cap D| \le \mu + \nu - k$ the points with labels in $B \cap D$ are also in general position within U_B. Therefore, for each element in $B \setminus D = B \cap C$ we can pick a point in $U_B \setminus (U_B \cap U_D)$ such that all the points with labels in B are in general position within U_B. Similarly, we can pick points for the elements of $D \cap A$ in $U_D \setminus (U_B \cap U_D)$ such that the points with labels in D are in general position within U_D. Without loss of generality, we can assume that the points with labels in B and the points with labels in D are in relative general position as subsets of $U_B + U_D = \mathbb{R}^{k-1}$.

For the remaining elements in $A \cap C = [n] \setminus (B \cup D)$ we can pick points in $\mathbb{R}^{k-1} \setminus (U_B \cup U_D)$ such that the points with labels in $A \cap C$ are in general position and the family of sets of points with labels in B, D, and $A \cap C$, respectively, is in relative general position. By Lemma 2.53, the matroid generated by this point set has the desired property.

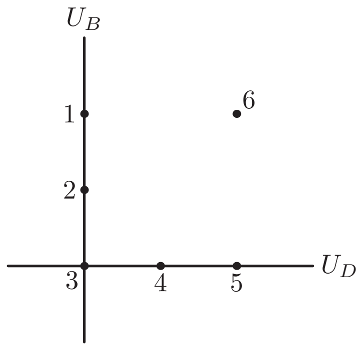

FIGURE 2.7. Non-split matroid constructed from two compatible splits in $\Delta(3,6)$ as in Example 2.60.

EXAMPLE 2.60. We continue our Example 2.42, where $k = 3$ and $n = 6$, considering the compatible splits $(\{1,2,6\},\{3,4,5\};2)$ and $(\{4,5,6\},\{1,2,3\};2)$. In the notation used in Construction 2.59 we have $A = \{1,2,6\}$, $B = \{3,4,5\}$, $C = \{4,5,6\}$, $D = \{1,2,3\}$, and $\mu = \nu = 2$. Hence $A \cap C = \{6\}$, $A \cap D = \{1,2\}$, $B \cap C = \{4,5\}$, and $B \cap D = \{3\}$. The matroid from Construction 2.59 is displayed in Figure 2.7. The non-split matroid polytope constructed in the proof of Theorem 2.58 has the f-vector $(18, 72, 101, 59, 14)$.

For the special case $k = 2$ the structure of the tropical Grassmannian and the Dressian is much simpler. The following proposition follows from [**84**, Theorem 3.4] in connection with Theorem 2.49.

PROPOSITION 2.61. *The tropical Grassmannian* $\mathrm{Gr}'''(2, n)$ *equals* $\mathrm{Dr}'(2, n)$*, and it is a simplicial complex which is isomorphic to the split complex* $\mathrm{Split}(\Delta(2, n))$*.*

Let us revisit the two smallest cases: The tropical Grassmannian $\mathrm{Gr}'''(2, 4)$ consists of three isolated points corresponding to the three splits of the regular octahedron, and $\mathrm{Gr}'''(2, 5)$ is a one-dimensional simplicial complex isomorphic to the Petersen graph; see Figure 2.5.

PROPOSITION 2.62. *The rays in* $\mathrm{Dr}(k, n)$ *correspond to the coarsest regular matroid subdivisions of* $\Delta(k, n)$*.*

PROOF. By definition, a ray in $\mathrm{Dr}(k, n)$ defines a regular matroid subdivision which is coarsest among the matroid subdivisions of $\Delta(k, n)$. We have to show that this is a coarsest among all subdivisions.

To the contrary, suppose that Σ is a coarsest matroid subdivision which can be coarsened to a subdivision Σ'. By construction the 1-skeleton of Σ' is contained in the 1-skeleton of Σ. From Theorem 2.51 it follows that Σ' is matroidal. This is a contradiction to Σ being a coarsest matroid subdivision. □

EXAMPLE 2.63. In view of Proposition 2.61, the first example of a tropical Grassmannian that is not covered by the previous results is the case $k = 3$ and $n = 6$. So we want to describe how the split complex $\mathrm{Split}(\Delta(3, 6))$ is embedded into $\mathrm{Gr}'''(3, 6)$. We use the notation of [**84**, Section 5]; see also [**81**, Section 4.3].

The tropical Grassmannian $\mathrm{Gr}'''(3, 6)$ is a pure three-dimensional simplicial complex which is not a flag complex. Its f-vector reads $(65, 550, 1395, 1035)$, and its homology is concentrated in the top dimension. The only non-trivial (reduced) homology group (with integral coefficients) is $H_3(\mathrm{Gr}'''(3, 6); \mathbb{Z}) = \mathbb{Z}^{126}$.

The splits with $A = \{1\} \cup A_1$, $\mu = 1$, and $A = \{1\} \cup A_3$, $\mu = 2$, are the 15 vertices of type "F". The splits with $A = \{1\} \cup A_2$ and $\mu \in \{1, 2\}$ are the 20 vertices of type "E". Here A_m is an m-element subset of $\{2, 3, \ldots, n\}$. The remaining 30 vertices are of type "G", and they correspond to coarsest subdivisions of $\Delta(3, 6)$ into three maximal cells. Hence they do not occur in the split complex. See also Billera, Jia, and Reiner [**9**, Example 7.13].

The 100 edges of type "EE" and the 120 edges of type "EF" are the ones induced by compatibility. Since $\mathrm{Split}(\Delta(3, 6))$ does not contain any "FF"-edges it is not an induced subcomplex of $\mathrm{Gr}'''(3, 6)$. The matroid shown in Figure 2.7 arises from an "EE"-edge.

The split complex is three-dimensional and has f-vector $(35, 220, 360, 30)$. It is not pure: The 30 maximal faces of dimension three are the tetrahedra of type "EEEE". The remaining 240 maximal faces are "EEF"-triangles.

The integral homology of $\mathrm{Split}(\Delta(3, 6))$ is concentrated in dimension two, and it is free of degree 144.

REMARK 2.64. Example 2.63 and Proposition 2.61 show that the split complex is a subcomplex of $\mathrm{Gr}'''(k, n)$ if $d = 2$ or $n \leq 6$. However, this does not hold in general: Consider the weight functions w, w' defined in the proof of [**84**, Theorem 7.1]. It is easily seen from Proposition 2.41 that w and w' are sums of the weight functions of compatible systems of vertex splits for $\Delta(3,7)$. Yet in the proof of [**84**, Theorem 7.1], it is shown that $w, w' \notin \mathrm{Gr}'''(3,7)$ for fields with characteristic not equal to two and equal to two, respectively.

2.8. Open Questions and Concluding Remarks

We showed that special split complexes of polytopes (e.g., of the polygons and of the second hypersimplices) already occurred in the literature albeit not under this name. So the following is natural to ask.

QUESTION 2.65. What other known simplicial complexes arise as split complexes of polytopes?

The split hyperplanes of a polytope define an affine hyperplane arrangement. For example, the coordinate hyperplane arrangements arises as the split hyperplane arrangement of the cross polytopes; see Example 2.33.

QUESTION 2.66. Which hyperplane arrangements arise as split hyperplane arrangements of some polytope?[†]

Jonsson [**57**] studies generalized triangulations of polygons; this has a natural generalization to simplicial complexes of split systems such that no $k+1$ splits in such a system are totally incompatible. See also [**76, 22**].

QUESTION 2.67. How do such *incompatibility complexes* look alike for other polytopes?

All computations with polytopes, matroids, and simplicial complexes were done with `polymake` [**37**]. The visualization also used `JavaView` [**77**].

We are indebted to Bernd Sturmfels for fruitful discussions. We also thank Hiroshi Hirai and an anonymous referee for several useful comments.

[†] For a study of hyperplane arrangements and an answer to a slightly different question see Section 3.4.

CHAPTER 3

More About Splits

In this chapter, we will continue to develop the theory of tight spans and splits of convex polytopes. First, we will give a collection of results about tight spans, splits, and (weak) compatibility some of which will be used later. In Section 3.2, we will examine some examples by discussing the splits and their compatibility for cubes, products of simplices, and products of hypersimplices, therewith giving partial computations of the split complexes of these polytopes. We conclude our general discussion about split theory in the third section by describing how splits can behave under some standard constructions for polytopes.

In Section 3.4, the connection of splits with hyperplane arrangements we pointed out in Section 2.8 will be further investigated. We show that each affine hyperplane arrangement occurs as the hyperplane arrangement of some weakly compatible split system for some polytope.

The last two sections are devoted to the extension of the theory of splits to oriented matroids and to the attempt to give a combinatorial description of splits.

3.1. Conditions for Splits and Compatibility

We have seen in Section 2.3, especially in Observation 2.12, some conditions when a hyperplane H defines a split of a polytope P. We will summarize and extend this in the following proposition, which can be derived directly from the definition of split and Observation 2.12.

Proposition 3.1. *For a hyperplane H and a polytope P with $H \cap \operatorname{relint} P \neq \emptyset$ the following are equivalent:*

(a) H induces a split on P,
(b) H meets all edges of P in a face of P,
(c) H meets all faces of P in a face of P or induces a split on them,
(d) H meets all facets of P in a face of P or induces a split on them,
(e) all vertices of the subdivision of P with maximal faces $P \cap H_+$ and $P \cap H_-$ are vertices of P,
(f) $H \cap P = \operatorname{conv}(\operatorname{Vert} P \cap H)$.

Here H_+ and H_- denote the two halfspaces in which $\mathbb{R}^{d+1}$ is divided by the hyperplane H.

The following specialization of Proposition 3.1 will be useful when constructing splits of concrete $0/1$-polytopes.

COROLLARY 3.2. *Let $P \subset \mathbb{R}^{d+1}$ be a $0/1$-polytope and H a hyperplane defined by*

$$\sum_{i \in I} x_i = k\,,$$

where $I \subseteq [d+1]$, $k \in \mathbb{Z}$. If $H \cap \operatorname{relint} P \neq \emptyset$ and for all edges (v,w) of P we have $\big| |\{i \in [d+1] \mid v_i = 1\}| - |\{i \in [d+1] \mid \bar{v}_i = 1\}| \big| \leq 1$, then H induces a split on P.

PROOF. Suppose we have a hyperplane H with these properties that does not define a split. By Proposition 3.1, this means that there exists some edge $(v, \bar{v})$ of P with $\sum_{i \in I} v_i = l > k$ and $\sum_{i \in I} \bar{v}_i = m < k$. Since the vertices are integral we must have $l, m \in \mathbb{Z}$ and hence $l \geq k+1, m \leq k-1$. This implies $\sum_{i \in I}(v_i - \bar{v}_i) \geq k+1-(k-1) = 2$, but on the other hand, the sum $\sum_{i \in I}(v_i - \bar{v}_i)$ can change by at most 1 since P is a $0/1$-polytope, and $\big| |\{i \in [d+1] \mid v_i = 1\}| - |\{i \in [d+1] \mid w_i = 1\}| \big| \leq 1$. This is the desired contradiction. □

The following is a direct corollary of Remark 2.13 (see also the discussion in Section 3.5) since affinely isomorphic polytopes have the same oriented matroid. However, we give a direct argument that does not use the language of oriented matroids. Note that – in contrast – combinatorially isomorphic polytopes need not to have the same splits for $d \geq 3$; see Remark 2.14.

COROLLARY 3.3. *Affinely isomorphic polytopes have the same splits.*

PROOF. This follows from Condition (f) of Proposition 3.1 since for a polytope P a hyperplane H, and an isomorphism φ the condition $H \cap P = \operatorname{conv}(\operatorname{Vert} P \cap H)$ directly leads to $\varphi(H) \cap \varphi(P) = \operatorname{conv}(\operatorname{Vert} \varphi(P) \cap \varphi(H))$ since φ maps hyperplanes to hyperplanes and preserves convexity. □

A key in understanding when two splits of a polytope P are (weakly) compatible is to examine the intersection of one of the split hyperplanes H_S with P. We call $P_S := P \cap H_S$ the *polytope induced by the split S on P*.

EXAMPLE 3.4.
(a) The polytopes induced by splits of the hypersimplex $\Delta(k,n)$ are products of hypersimplices. The split S induced by the $(A,B;\mu)$-hyperplane is isomorphic to $\Delta(k-\mu, |A|) \times \Delta(\mu, |B|)$. If S is a vertex split we have $|A| = k$ and $\mu = k-1$ so we get the product of a $(k-1)$-dimensional and an $(n-k-1)$-dimensional simplex. Also in the case $k = 2$ all polytopes induced by splits are products of simplices.
(b) The polytopes induced by splits of the n-cube C_n are products of cubes and hypersimplices. The split S defined by the hyperplane (3.1) of Proposition 3.15 below is isomorphic to $\Delta((|I|-k)/2, |I|) \times C_{n-|I|}$.
(c) If S_v is a vertex split of a polytope P, the polytope induced by S_v is isomorphic to the vertex figure of P at v.

Now we can give conditions for two splits of a polytope to be compatible or weakly compatible:

PROPOSITION 3.5. *Let S_1 and S_2 be two splits of a polytope P.*

(a) The splits S_1 and S_2 are compatible if and only if $P_{S_1} \cap P_{S_2} = P_{S_1} \cap H_{S_2}$ is a face of P_{S_1} (possibly empty).

(b) The splits S_1 and S_2 are weakly compatible if and only if either $P_{S_1} \cap H_{S_2}$ is a face of P_{S_1} or H_{S_2} induces a split on P_{S_1}.

PROOF. (a) The splits S_1 and S_2 are compatible if and only if H_{S_1} and H_{S_2} do not meet in $\operatorname{relint} P$. It is clear that this is equivalent to $\operatorname{relint} P_{S_1} \cap H_{S_2} = \emptyset$. However, since S_2 is a split this means that $P_{S_1} \cap H_{S_2}$ is a face of P_{S_1}.

(b) The splits S_1 and S_2 are weakly compatible if and only if the common refinement Σ of the two subdivisions S_1 and S_2 does not use additional vertices. Obviously, these vertices can only occur in $P_{S_1} \cap H_{S_2}$ what happens if and only if H_{S_2} does not induce a split on P_{S_1}.

□

The following more general condition can be used subsequently to construct compatible or weakly compatible split systems.

PROPOSITION 3.6. *Let $\mathcal{S}$ be a split system and S another split of a polytope P.*

(a) If $\mathcal{S}$ is compatible then $\mathcal{S} \cup \{S\}$ is compatible if and only if $P_{S'} \cap H_S$ is a face of $P_{S'}$ for all $S' \in \mathcal{S}$.

(b) If $\mathcal{S}$ is weakly compatible then $\mathcal{S} \cup \{S\}$ is weakly compatible if and only if for all faces F of $\Sigma_{\mathcal{S}}(P)$ either $H_S \cap F$ is a face of F or H_S induces a split on F.

PROOF. (a) Follows inductively from Proposition 3.5 (a).

(b) The split system $\mathcal{S} \cup \{S\}$ is weakly compatible if and only if the common refinement of $\Sigma_{\mathcal{S}}(P)$ and S is again a subdivision of P without new vertices. This means that for all faces F of $\Sigma_{\mathcal{S}}(P)$ which meet H_S in the interior we must have $H_S \cap F = \operatorname{conv}(\operatorname{Vert}(F) \cap H)$. But, by Proposition 3.1, this means that $H_S \cap F$ has to be a face of F, or H_S has to induce a split on F.

□

For the weakly compatible case, this leads us to the following list of conditions (cf. Proposition 3.1).

COROLLARY 3.7. *Let P be a polytope, $\Sigma_{\mathcal{S}}(P)$ a subdivision of P induced by a weakly compatible set of split $\mathcal{S}$, and $S \notin \mathcal{S}$ a split of P. Then the following are equivalent:*

(a) $\mathcal{S} \cup \{S\}$ is weakly compatible,
(b) H_S meets all edges of $\Sigma_{\mathcal{S}}(P)$ in a face of $\Sigma_{\mathcal{S}}(P)$,
(c) H_S meets all faces of $\Sigma_{\mathcal{S}}(P)$ in a face of $\Sigma_{\mathcal{S}}(P)$ or induces a split on them,
(d) H_S meets all codimension-one-faces of $\Sigma_{\mathcal{S}}(P)$ in a face of $\Sigma_{\mathcal{S}}(P)$ or induces a split on them,
(e) $H_S \cap C = \operatorname{conv}(\operatorname{Vert} C \cap H_S)$ for all maximal cells C of $\Sigma_{\mathcal{S}}(P)$.

PROOF. That (a) is equivalent to (c) follows directly from Proposition 3.6 (b). Suppose now that there is some face F of $\Sigma_{\mathcal{S}}(P)$ which is not met by H_S in a face or in which H_S does not induce a split. By Proposition 3.1, this means that there is an edge of F and hence of $\Sigma_{\mathcal{S}}(P)$ which is not met in a face by H_S. This shows the equivalence of (a) and (b). The equivalence of (a) and (d) follows by a slight strengthening of the argument in the proof of Proposition 3.6 (b). That (e) is equivalent to the other conditions is obvious. □

Note that, in fact, a hyperplane H which fulfills one of the conditions (b) to (e) of Corollary 3.7 defines a split S on P, so this does not have to be checked a priori.

If $\mathcal{S}$ is actually compatible, then checking weak compatibility of $\mathcal{S} \cup \{S\}$ can be reduced to checking pairwise weak compatibility.

LEMMA 3.8. *Let P be a polytope, $\mathcal{S}$ a compatible split system of P, and S another split of P. Then the split system $\mathcal{S} \cup \{S\}$ is weakly compatible if and only if for all $S' \in \mathcal{S}$ the splits S and S' are weakly compatible.*

PROOF. We only have to prove that the condition is sufficient. So suppose that $\mathcal{S} \cup \{S\}$ is not weakly compatible. Then, by Proposition 3.6 (b), there exists some face F of $\Sigma_{\mathcal{S}}(P)$ with $H_S \cap F \neq \emptyset$ on which H_S does not define a split. But since $\mathcal{S}$ is compatible, F is the face of $H_{S'} \cap P$ for some $S' \in \mathcal{S}$; and hence S and S' are not weakly compatible. □

This leads to the following interesting consequence.

COROLLARY 3.9. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two compatible split systems of a polytope P. If each two elements $S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$ are weakly compatible, then $\mathcal{S}_1 \cup \mathcal{S}_2$ is weakly compatible.*

PROOF. Suppose $\mathcal{S}_1 \cup \mathcal{S}_2$ is not weakly compatible. Then in the common refinement of the subdivision $\Sigma_{\mathcal{S}_1}(P)$ and $\Sigma_{\mathcal{S}_2}(P)$ there has to be an additional vertex v. Let F be the inclusion-minimal face of $\Sigma_{\mathcal{S}_1}(P)$ with $v \in F$, and $\bar{\mathcal{S}} := \{S \in \mathcal{S}_2 \mid v \in H_S\}$. If $|\bar{\mathcal{S}}| \leq 1$ or $H_S \cap F = H_{S'} \cap F$ for all $S, S' \in \bar{\mathcal{S}}$ we get a contradiction by Lemma 3.8. So we have $v \in I := \bigcap_{S \in \bar{\mathcal{S}}} S$ which implies $I \cap \operatorname{relint} P \neq \emptyset$. But this contradicts the compatibility of $\mathcal{S}_2$. □

REMARK 3.10. A set of vertex splits of a polytope P as in Remark 2.14 is compatible if and only if it is weakly compatible, if and only if their corresponding vertices are not connected by an edge of the polytope. Hence a set of those splits is weakly compatible if and only if it is compatible. So the simplicial complex of stable sets of the edge graph of a polytope is a subcomplex of $\mathrm{Split}(P)$ and $\mathrm{SecFan}'(P)$; see Corollary 7.9 and Theorem 7.10 for an application of this observation.

If one has a (weakly) compatible set $\mathcal{S}_1$ of such splits and a set $\mathcal{S}_2$ of other splits of P, the set $\mathcal{S}_1 \cup \mathcal{S}_2$ is a weakly compatible split system of P if and only if $\mathcal{S}_2$ is a weakly compatible split system for the polytope $\bar{P}$ which arises from P by cutting of all vertices corresponding to the splits in $\mathcal{S}_1$. Especially all splits of $\bar{P}$ are splits of P, namely those splits S for which $\mathcal{S}_1 \cup \{S\}$ is (weakly) compatible.

If P is a simple polytope, there exist vertex splits for all vertices; see Remark 2.14. So a stable set of the vertex graph of P gives us a compatible split system. As an example consider the n-dimensional cube C_n. Taking the set of all vertices with an even number of ones which is easily seen to be a stable set of the vertex graph of the cube and cutting them of one arrives at a polytope $\bar{C}_n$ with 2^{n-1} vertices, half of the vertices of C_n. By looking for splits or split subdivisions of $\bar{C}_n$ one gets splits or split subdivisions of the cube.

In Section 2.3, we defined the split polyhedron of P, whose facets correspond to the splits of P. On the other hand, there is (in general) no correspondence of split triangulations or maximal weakly compatible split systems with vertices $\text{SplitPoly}(P)$. This follows directly from the fact that there exists split triangulations which do not use the maximal possible number of splits and hence cannot define a vertex. For an example consider the triangulation in Figure 2.3 (left) of the 3-cube which is the refinement of three splits. However, $\text{SplitPoly}(C_3)$ has dimension four; see also Example 2.34. However, at least the following weaker statement holds.

Proposition 3.11. *Let $\mathcal{S}$ be a weakly compatible split system of a polytope P. Then the facets of* $\text{SplitPoly}(P)$ *corresponding to the splits in $\mathcal{S}$ share a common vertex.*

Proof. Let F be the face of $\text{SecPoly}(P)$ obtained as the intersection of all facets corresponding to splits in $\mathcal{S}$. Just as well, we define F' as the face of $\text{SplitPoly}(P)$ that is the intersection of the facets of $\text{SplitPoly}(P)$ corresponding to splits in $\mathcal{S}$. By the definition of $\text{SplitPoly}(P)$, we have $F \subset F'$. So any vertex of F' works. □

So the split polyhedron may be used to find all maximal weakly compatible split systems (and all split triangulations): For each vertex (or inclusion minimal face) of $\text{SplitPoly}(P)$ take the splits corresponding to all its facets and try which one are weakly compatible.

By considering all split triangulations of a polytope P, one can also define the *inner split polytope* $\text{InSplitPoly}(P)$ of P is the convex hull of all vertices of the secondary polytope of P which are split triangulations. In contrast to the outer approximation $\text{SplitPoly}(P)$, the polytope $\text{InSplitPoly}(P)$ gives us an inner approximation of $\text{SecPoly}(P)$. This can be used to give bounds for some properties of $\text{SecPoly}(P)$, for example the volume. (Note, however, that $\text{InSplitPoly}(P)$ may well be empty.)

Weakly compatible split systems which are at the same time totally incompatible are interesting because they correspond to central hyperplane arrangements and because their tight spans are zonotopes; see Section 3.4. The following gives a necessary condition for a split system to have this property.

Lemma 3.12. *Let P be a d-dimensional polytope and $\mathcal{S}$ a totally incompatible split system for P. If $\mathcal{S}$ is weakly compatible, then* $\left|\bigcap_{S\in\mathcal{S}} H_S \cap \operatorname{Vert} P\right| \geq d+1-|\mathcal{S}|$.

Proof. The intersection $I := \bigcap_{S\in\mathcal{S}} H_S$ has codimension at most $|\mathcal{S}|$. Since $\mathcal{S}$ is totally incompatible, I meets the interior of P, and so $I \cap P$ is a polytope of dimension at least $(d-|\mathcal{S}|)$ and has at least $d+1-|\mathcal{S}|$ vertices. All these vertices has to be vertices of P by Proposition 3.1. This shows the claim. □

For the special case $|\mathcal{S}| = 2$ this can be reformulated as follows:

Corollary 3.13. *Let S_1 and S_2 be two splits of a d-dimensional polytope P that are weakly compatible but not compatible. Then we have $|H_{S_1} \cap H_{S_2} \cap \operatorname{Vert} P| \geq d-1$.*

We close this section with a simple bound on the dimension of the tight span of the coherent sum of two weight functions.

PROPOSITION 3.14. *Let w_1, w_2 be lifting functions of a polytope P and $\mathcal{T}_{w_1}(P)$, $\mathcal{T}_{w_2}(P)$ the corresponding tight spans. If (w_1, w_2) is coherent, we have*

$$\dim \mathcal{T}_{w_1+w_2}(P) \leq \dim \mathcal{T}_{w_1}(P) + \dim \mathcal{T}_{w_2}(P)\,.$$

PROOF. Let F be a face of $\mathcal{T}_{w_1+w_2}(P)$ of dimension $d := \dim \mathcal{T}_{w_1+w_2}(P)$. Each face F of the Minkowski sum of two polyhedra is the Minkowski sum of two faces F_1, F_2, one from each summand. So we have faces $F_1 \subset \mathcal{T}_{w_1}(P)$ and $F_2 \subset \mathcal{T}_{w_2}(P)$ such that $F = F_1 + F_2$. This implies that $d = \dim F \leq \dim F_1 + \dim F_2 \leq \dim \mathcal{T}_{w_1}(P) + \dim \mathcal{T}_{w_2}(P)$. □

3.2. Examples

In Section 2.5, we discussed the splits and their compatibility relations for hypersimplices. In this section, we compute the splits of other families of polytopes and the compatibility relations for some of them.

The first polytope we discuss is the n-dimensional cube C_n. To get a symmetric description of the splits, we consider the cube as $C_n := [-1, 1]^n$.

PROPOSITION 3.15. *The splits of the n-dimensional cube C_n are given by hyperplanes defined as*

$$\sum_{i \in I} \epsilon_i x_i \;=\; k\,, \tag{3.1}$$

for some $I \subset [n]$ where $\epsilon_i \in \{-1, 1\}$, $0 \leq k \leq |I| - 2$, and $k \equiv |I| \mod 2$.

PROOF. For $n = 2$ we have the two splits defined by $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$. So the proposition is obviously true, and we can proceed inductively.

First, we prove the existence of the splits. For $I \subsetneq [n]$ we look at the face of C_n defined by $x_i = 1$ for all $i \notin I$. This face F is isomorphic to $C_{|I|}$. It is easily seen that the splits of F extends to splits of C_n, so we can assume that $I = [n]$. Obviously, for each vertex v of C_n we have $\sum_{i=1}^{n} \epsilon_i v_i \equiv n \mod 2$, and the sum changes by ± 2 if we change one v_i form $+1$ to -1 or vice versa, which corresponds to going along an edge of C_n. So the two vertices of an edge of C_n cannot be on different sides of our splitting hyperplane. Furthermore, one can compute that the hyperplane (3.1) meets the interior of C_n if and only if $-(|I| - 2) \leq k \leq |I| - 2$ and $k \equiv |I| \mod 2$. This shows that all those hyperplanes define splits by Proposition 3.1.

To prove that these are the only splits we proceed as follows: Let a split of C_n be given by the hyperplane $H := \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i x_i = \beta\}$ for some $\alpha_i, \beta \in \mathbb{R}$. If $\alpha_i = 0$ for some i, the hyperplane must also define a split on the facet of C_n defined by $x_i = 1$ which is isomorphic to C_{n-1}, and we are done by induction. So we can assume that for all i we have $\alpha_i \neq 0$. Let now H meet a facet F of C_n in the interior; without loss of generality we assume that $F = C_n \cap \{x \in \mathbb{R}^n \mid x_1 = 1\}$. Then, by Proposition 3.1, the equation $\sum_{i=2}^{n} \alpha_i x_i = \beta - 1$ has to define a split on $F \cong C_{n-1}$. So we have $\alpha_i = \pm\lambda$ for all $i > 1$ and $\beta = \lambda\bar{\beta} + 1$ for some $\lambda \in \mathbb{R}$, $0 \leq \bar{\beta} \leq n - 3$, $\bar{\beta} \equiv n - 1 \mod 2$. Similar equations also have to hold for another $j \in [n]$, so we get that $\alpha_i = \pm\lambda = \pm\lambda'$ for $i \notin \{1, j\}$ and that $\beta = \lambda\bar{\beta} + 1 = \lambda'\bar{\beta}' + 1$. By setting $\lambda := \lambda' := 1$, we get the desired result $\alpha_i = \pm 1$ and $0 \leq \beta \leq n - 2$, $\beta \equiv n \mod 2$. □

COROLLARY 3.16. *The n-dimensional cube C_n has*

$$3^{n-1}n - \frac{3^n - 1}{2} \tag{3.2}$$

splits which come in $\lfloor n^2/4 \rfloor$ equivalence classes.

PROOF. All splits are given by Proposition 3.15. We sum over all possible cardinalities for I, which are $2, \dots, n$ (since $0 \leq k \leq |I| - 2$ implies $|I| \geq 2$). For a fixed cardinality $|I| = l$ we can choose the ϵ_i in 2^l ways and have $\lfloor l/2 \rfloor$ different possibilities for k in $0 \leq k \leq l - 2$, $k \equiv l \mod 2$. This gives a total of $\sum_{l=2}^{n} \left\lfloor \frac{l}{2} \right\rfloor 2^l \binom{n}{l}$ different equations of the form (3.1). However, if $k = 0$, multiplying all ϵ_i by -1 does not change the split, so we have to subtract half of the splits for $k = 0$. This leads to a total of

$$\sum_{l=2}^{n} \left\lfloor \frac{l}{2} \right\rfloor 2^l \binom{n}{l} - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} 2^{2m-1} \binom{n}{2m}$$

splits. This can be shown to equal (3.2).

It is clear that two of the splits are equivalent if and only if they have the same cardinality for I and the same k. So there are $\sum_{l=2}^{n} \left\lfloor \frac{l}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor$ equivalence classes. □

We now want to examine when two splits of the cube are compatible. According to Proposition 3.15, a split of the cube is given by some set $\emptyset \neq I \subset [n]$, some vector $\epsilon \in \{+1, -1\}^I$, and some $0 \leq k \leq |I| - 2$. We define the sets $I_+ := \{i \in I \mid \epsilon(i) = +1\}$, $I_- := \{i \in I \mid \epsilon(i) = -1\}$, and $I_0 = [n] \setminus I$; and we will abbreviate this split by (I_+, I_-, k).

If we now take two splits $S_1 = (I_+, I_-, k)$ and $S_2 = (J_+, J_-, l)$ of C_n, we can partition $I \cup J$ into the three parts $E := \{i \in I \cup J \mid \epsilon_1(i) = \epsilon_2(i)\}$, $D := \{i \in I \cup J \mid \epsilon_1(i) = -\epsilon_2(i)\}$, and $Z := (I \cup J) \setminus (E \cup D) = (I \setminus J) \cup (J \setminus I)$. With this notions, we can formulate the following compatibility condition.

PROPOSITION 3.17. *Let $S_1 = (I_+, I_-, k)$ and $S_2 = (J_+, J_-, l)$ be two splits of the the cube C_n. Then S_1 and S_2 are compatible if*

$$2|E| + |Z| \leq k + l \quad \text{or} \tag{3.3}$$

$$2|D| + |Z| \leq |k - l|\,. \tag{3.4}$$

PROOF. According to the definition, the splits S_1 and S_2 are compatible if and only if there does not exist a point $x \in \operatorname{int} C_n = (-1, 1)^n$ such that

$$\begin{aligned} \sum_{i \in I_+} x_i - \sum_{i \in I_-} x_i &= k \quad \text{and} \\ \sum_{i \in J_+} x_i - \sum_{i \in J_-} x_i &= l\,. \end{aligned} \tag{3.5}$$

For the rest of this proof, we abbreviate $I_{\alpha\beta} = I_\alpha \cap J_\beta$ for $\alpha, \beta \in \{+, -, 0\}$, define $c := |I_{+0}| + |I_{0+}| + |I_{0-}| + |I_{-0}|$, and assume without loss of generality that $\min(k, l) = l$. By adding and subtracting the Equations (3.5), we get the following system of equations, which is equivalent to (3.5).

$$2\sum_{i\in I_{++}} x_i + \sum_{i\in I_{+0}\cup I_{0+}} x_i - 2\sum_{i\in I_{--}} x_i - \sum_{i\in I_{-0}\cup I_{0-}} x_i \;=\; k+l \quad \text{and}$$
$$2\sum_{i\in I_{+-}} x_i + \sum_{i\in I_{+0}\cup I_{0-}} x_i - 2\sum_{i\in I_{-+}} x_i - \sum_{i\in I_{-0}\cup I_{0+}} x_i \;=\; k-l\,. \tag{3.6}$$

Now one sees that for the existence of a point $x \in (-1,1)^n$ that satisfies the first of these equations, it is necessary that

$$2(|I_{++}| + |I_{--}|) + c > k + l\,. \tag{3.7}$$

Just as well, we get that for the existence of a point $x \in (-1,1)^n$ that satisfies the second equation of (3.6) it is necessary that

$$2(|I_{+-}| + |I_{-+}|) + c > k - l\,. \tag{3.8}$$

(Recall that we assumed that $k \geq l$.) This finishes the proof.

□

The second class of polytopes we will look at are products of two simplices. The regular subdivisions of these polytopes are especially interesting since since their tight spans are tropical polytopes; see Remark 2.6. We consider the k-simplex $\Delta_k \subseteq \mathbb{R}^k$ as $\operatorname{conv}\{0, e_1, \ldots, e_k\}$.

Proposition 3.18. *The splits of the product of simplices $P := \Delta_k \times \Delta_l$ are given by the hyperplanes defined by*

$$\sum_{i\in A} x_i - \sum_{j\in B} x_j \;=\; 0 \quad \textit{and} \quad \sum_{i\in A} x_i + \sum_{j\in B} x_j \;=\; 1\,, \tag{3.9}$$

where A and B are non-empty subsets of $[k]$ and $[k+l]\setminus[k]$, respectively.

Proof. First we remark that for all non-empty $A \subseteq [k]$, $B \subseteq [k+l]\setminus[k]$ the point x defined by

$$x_i \;=\; \begin{cases} \frac{1}{2|A|}, & \text{if } i \in A\,,\\ \frac{1}{4(k-|A|)}, & \text{if } i \in [k]\setminus A\,,\\ \frac{1}{2|B|}, & \text{if } i \in B\,,\\ \frac{1}{4(l-|B|)}, & \text{if } i \in [k+l]\setminus([k]\cup B)\,,\end{cases}$$

is in the interior of P since $x_i > 0$ for all i and $\sum_{i\in[k]} x_i, \sum_{i\in[k+l]\setminus[k]} x_i < 1$; and in both hyperplanes defined by the Equations (3.9), since $\sum_{i\in A} x_i = 1/2$, $\sum_{i\in B} x_i = 1/2$. Hence all those hyperplanes meet the interior of P.

It is easily seen that two vertices $u = (x_1, x_2)$, $v = (y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^l$ of $\Delta_k \times \Delta_l$ are connected by an edge if and only if $x_1 = y_1$ or $x_2 = y_2$. So the second of the Equations (3.9) is a split by Corollary 3.2. Since only one of the sums in the first equation changes by going from u to v, it can be seen as in the proof of Corollary 3.2 that this equation is also a split.

Let now $H = \{x \in \mathbb{R}^{k+l} \mid \sum_{i=1}^{k+l} \alpha_i x_i = \beta\}$ define a split of $\Delta_k \times \Delta_l$ for some $\alpha_i, \beta \in \mathbb{R}$. We can assume without restriction that the first non-zero α_i is equal to 1. However, since the matrix of vertices of a product of simplices is totally unimodular, all other non-zero α_j has to be equal to ± 1. By Lemma 3.27 below, there exists some vertex v of Δ_l such that H defines a split or a non-trivial face on $\Delta_k \times \{v\}$. Since simplices have no splits, H has to define a face of this facet. So we can conclude that we cannot have $\alpha_i = -\alpha_j$ for $i, j \in [k]$ since H then would meet the interior of $\Delta_k \times \{v\}$. The only remaining possibilities for H are therefore the Equations (3.9) for arbitrary A or B. Since for empty A or B those hyperplanes would not cut the interior of $\Delta_k \times \Delta_l$, the proof is complete. □

REMARK 3.19. (a) The first part of the proof shows that the splits defined by the two hyperplanes (3.9) for the same A and B are incompatible.

(b) By considering $\Delta(1, k+1) = \mathrm{conv}\{e_1, \dots, e_{k+1}\} \subset \mathbb{R}^{k+1}$ instead of Δ_k as the standard simplex, one can write all splits of $\Delta_k \times \Delta_l \cong \Delta(1, k+1) \times \Delta(1, l+1)$ in the more symmetric form

$$\sum_{i \in A} x_i = \sum_{j \in B} x_j \tag{3.10}$$

for all non-trivial subsets $A \subseteq [k+1]$, $B \subseteq [k+l+2] \setminus [k+1]$. Note that taking the complements of A and B yields the same splits (but no other choice).

COROLLARY 3.20. *The product $\Delta_k \times \Delta_l$ of two simplexes has $2 \cdot (2^k - 1) \cdot (2^l - 1)$ splits which come in $\lceil k \cdot l/2 \rceil$ equivalence classes.*

PROOF. The splits are given by the Equations (3.9). Since we can choose arbitrary nonempty sets $A \subsetneq [k]$ and $B \subsetneq [k+l] \setminus [k]$ and combine them in each way we get $(2^k - 1) \cdot (2^l - 1)$ splits of each type. This shows the claim for the number of splits.

For the proof of the second part, we use the alternative characterization of the splits in Remark 3.19 (b). Consider two nontrivial subsets $A, A' \subset [k+1]$, $B, B' \subset [k+l+2] \setminus [k+1]$. If $|A| = |A'|$ and $|B| = |B'|$, or $|A| = k+1-|A'|$ and $|B| = l+1-|B'|$ the splits are equivalent. So we get $kl/2$ equivalence classes, except in the case in which both k and l are odd, where we have $(kl+1)/2$ classes. □

Also for the product of two simplices, we want to give a condition when two splits are compatible. According to Remark 3.19 (b), a splits of the product of a k- and an l-dimensional simplex is defined by two sets $\emptyset \neq A \subsetneq [k+1]$ and $\emptyset \neq B \subsetneq [k+l+2] \setminus [k+1]$ and this representation is unique up to simultaneously taking the complements of A and B. We will write $S = (A, B)$ for a split defined this way.

PROPOSITION 3.21. *Let $S = (A, B)$ and $T = (C, D)$ be two splits of the product $\Delta(1, k+1) \times \Delta(1, l+1)$. Then S and T are compatible if and only if either*

$$\begin{aligned} A \subseteq C \quad &\textit{and} \quad && D \subseteq B, \quad \textit{or} \\ C \subseteq A \quad &\textit{and} \quad && B \subseteq D. \end{aligned} \tag{3.11}$$

PROOF. We define the sets

$$\begin{aligned} &A_1 = A \setminus C, && A_2 = C \setminus A, && A_3 = A \cap C, && A_4 = [k+1] \setminus (A \cup C), \\ &B_1 = B \setminus D, && B_2 = D \setminus B, && B_3 = B \cap D, && B_4 = [l+k+2] \setminus [k+1] \setminus (B \cup D); \end{aligned}$$

and

$$X_i = \sum_{i \in A_i} x_i, \quad Y_i = \sum_{i \in B_i} y_i .$$

Then the hyperplanes for S and T are defined by

$$X_1 + X_3 = Y_1 + Y_3 \quad \text{and} \quad X_2 + X_3 = Y_2 + Y_3\,, \tag{3.12}$$

respectively. If we subtract these two equations, we get

$$X_1 - X_2 = Y_1 - Y_2\,. \tag{3.13}$$

We will first proof that the conditions (3.11) are sufficient for the compatibility. So suppose S and T were not compatible, $A \subseteq C$, and $D \subseteq B$. This implies that for any $x \in H_S \cap H_T$ we have $X_1 = Y_2 = 0$. From this and Equation (3.13) we can further conclude that $X_2 = Y_1 = 0$. So $x_i = 0$ for all $i \in A_2 \cup B_1$. Since x should not be in the boundary, this implies that A_2 and B_1 are empty, and so $S = T$. The second case follows similarly.

For the necessity, assume that (3.11) does not hold. This is equivalent to

$$\begin{aligned} A_1 \neq \emptyset \quad &\text{or} \quad B_2 \neq \emptyset, \quad \text{and} \\ A_2 \neq \emptyset \quad &\text{or} \quad B_1 \neq \emptyset. \end{aligned} \tag{3.14}$$

We will now distinguish several cases, depending on how many of the sets A_j, B_j are empty. In each case we will give a point $x \in \operatorname{relint}(\Delta(1, k+1) \times \Delta(1, l+1)) \cap H_S \cap H_T$. This will be done by assigning values in $(0, 1)$ to all X_j, Y_j for which A_j, B_j, respectively, are non-empty such that (3.13) holds and $\sum X_j = \sum Y_j = 1$. The explicit coordinates of x are then obtained by setting $x_i = \frac{X_j}{|A|_j}$, $x_i = \frac{Y_j}{|B|_j}$, for $i \in A_j$, $i \in B_j$, respectively.

Case 1: Non of the sets A_j, B_j is empty. Then we simply set $X_j, Y_j = \frac{1}{4}$ for all $j \in \{1, 2, 3, 4\}$.

Case 2: One of the sets A_j, B_j is empty. We assume without loss of generality that $A_1 = \emptyset$. Then we set $X_3 = \frac{1}{2}$, $X_4 = Y_2 = \frac{3}{8}$, $Y_1 = Y_3 = \frac{1}{4}$, and $X_2 = Y_4 = \frac{1}{8}$.

Case 3: Two of the sets A_j, B_j are empty. As in Case 2, we assume that one of these sets is A_1. With our assumption (3.14), and taking into account that neither A, B, C, D nor their complements (in $[k+1]$ and $[k+l+2] \setminus [k+1]$, respectively) can be empty, we get the following possibilities:

- ▹ $A_1 = A_2 = \emptyset$: Set $X_3 = X_4 = \frac{1}{2}$ and $Y_i = \frac{1}{4}$ for all $i \in \{1, 2, 3, 4\}$.
- ▹ $A_1 = B_1 = \emptyset$: Set $X_3 = Y_3 = \frac{1}{2}$ and $X_2 = X_4 = Y_2 = Y_4 = \frac{1}{4}$.
- ▹ $A_1 = B_3 = \emptyset$: Set $X_4 = Y_2 = \frac{1}{2}$ and $X_2 = X_3 = Y_1 = Y_4 = \frac{1}{4}$.
- ▹ $A_1 = B_4 = \emptyset$: Set $X_3 = Y_2 = \frac{1}{2}$ and $X_2 = X_4 = Y_1 = Y_3 = \frac{1}{4}$.

Case 4: Three of the sets A_j, B_j are empty. We again assume that A_1 is one of the sets. It remain three possibilities:

- ▹ $A_1 = A_2 = B_3 = \emptyset$: Set $X_4 = \frac{2}{3}$ and $X_3 = Y_1 = Y_2 = Y_4 = \frac{1}{3}$.
- ▹ $A_1 = A_2 = B_4 = \emptyset$: Set $X_3 = \frac{2}{3}$ and $X_4 = Y_1 = Y_2 = Y_3 = \frac{1}{3}$.
- ▹ $A_1 = B_3 = B_4 = \emptyset$: Set $Y_2 = \frac{2}{3}$ and $X_2 = X_3 = X_4 = Y_1 = \frac{1}{3}$.

Case 5: Four of the sets A_j, B_j are empty. By assuming that A_1 is one of them, this yields $A_1 = A_2 = B_3 = B_4 = \emptyset$. We set $X_3 = X_4 = Y_1 = Y_2 = \frac{1}{2}$.

□

Example 3.22. Define $A_i := \{1, \ldots, i\}$ and $B_j := B_j = \{k+1, \ldots, k+j\}$. Then it can be shown that the set

$$\mathcal{S} := \left\{(A_i, B_j) \;\middle|\; 1 \le i \le k, 1 \le j \le l\right\}$$

of splits of $P := \Delta(1, k+1) \times \Delta(1, l+1)$ is weakly compatible. Since $|\mathcal{S}| = k \times l = |\mathrm{Vert}\, P| - \dim P - 1$, the split subdivision $\Sigma_{\mathcal{S}}(P)$ defined by $\mathcal{S}$ is a triangulation by Corollary 2.29.

This triangulation is called the staircase triangulation; see Billera, Cushman, and Sanders [**6**]. In fact, for any ordering of the vertices of Δ_k and any ordering of the vertices of Δ_l there exists such a split triangulation. See Figure 3.1 for examples of tight spans of staircase triangulations.

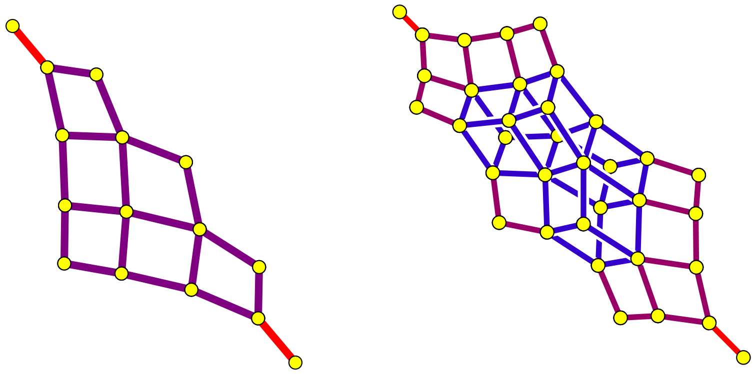

Figure 3.1. Tight spans of staircase triangulations of $\Delta_4 \times \Delta_2$ and $\Delta_4 \times \Delta_3$.

As a last example, we will examine the product $\Delta(k, n) \times \Delta(l, m)$ of two general hypersimplices.

Proposition 3.23. *The splits of the product of two hypersimplices* $\Delta(k, n) \times \Delta(l, m)$ *are given by the hyperplanes defined by*

$$\sum_{i \in A} x_i + \sum_{j \in B} x_j = \alpha\,, \tag{3.15}$$

where $A \subset [n]$, $B \subset [n+m] \setminus [n]$, $\alpha \in \mathbb{N}$ *and* $\max(0, k - (n - |A|)) + \max(0, l - (m - |B|)) < \alpha < \min(k, |A|) + \min(l, |B|)$.

Proof. We first examine when a hyperplane defined by Equation (3.15) meets the interior of $P := \Delta(k, n) \times \Delta(l, m)$. A hyperplane H meets the interior of a polytope if and only if there exist vertices of P on either side of H. In order to have a vertex v of P with $\sum_{i \in A} v_i + \sum_{j \in B} v_j > \alpha$, it is necessary and sufficient that $\alpha < \min(k, |A|) + \min(l, |B|)$ since this is the maximum of the left side of Equation (3.15) taken over all vertices of P. In the same manner, one sees that $\alpha > \max(0, k-(n-|A|)) + \max(0, l-(m-|B|))$ is necessary and sufficient for the existence of some vertex v of P with $\sum_{i \in A} v_i + \sum_{j \in B} v_j < \alpha$. This shows that a hyperplane defined by (3.15) meets the interior of P if and only if the condition of the proposition is satisfied.

It follows from Corollary 3.2 that the hyperplanes (3.15) define splits. So it remains to proof that all splits of P have to be of this form.

Let H be defined by $\sum_{i \in [n]} \alpha_i x_i + \sum_{i \in [m+n] \setminus [n]} \beta_i x_i = 0$ such that H defines a split of P. By Lemma 3.27 (b), for each vertex v of $\Delta(k, n)$ we get that H defines a split or a non-trivial face of $\{v\} \times \Delta(l,m)$. We can argue as in the proof of Proposition 2.39 that there exists at least one such v such that H actually defines a split; of course the same holds of one exchanges the role of the factors. So by our characterization of splits of hypersimplices in Proposition 2.39, this implies that H has to be of the form $\sum_{i\in A} x_i + \beta \sum_{j\in B} x_j = \alpha$ for some $\alpha, \beta \in \mathbb{R}$. By possibly exchanging the factors, we can assume that $\beta \geq 1$. However, then any edge from (e_I, e_J) to $(e_I, e_{J\cup\{i\}\setminus\{j\}})$ is met in the interior if we take $i \in B$ and $j \notin B$. □

3.3. Polytope Constructions and Splits

In this section, we will describe how splits behave under some well-known polytope constructions. A very simple construction is the *pyramid* $\operatorname{Pyr} P \subset \mathbb{R}^{d+1} \times \mathbb{R}$ of $P \subset \mathbb{R}^{d+1}$. It is derived as the convex hull of $P \times \{0\}$ and some point $v \in \mathbb{R}^{d+1} \times (\mathbb{R} \setminus \{0\})$. Likewise, the *bipyramid* $\operatorname{BiPyr} P \subset \mathbb{R}^{d+1} \times \mathbb{R}$ of P is defined as the convex hull of $P \times \{0\}$, some point $v \in \mathbb{R}^{d+1} \times \mathbb{R}_{>0}$, and some point $v' \in \mathbb{R}^{d+1} \times \mathbb{R}_{<0}$. For a simpler notation, we identify P with $P \times \{0\}$ and splitting hyperplanes H of P with $H \times \{0\}$.

Proposition 3.24. *Let P be a polytope and $\mathcal{S}$ the set of all splits of P.*

(a) *The splits of a pyramid $\operatorname{Pyr} P$ over P with additional vertex v are induced by the hyperplanes $H = \operatorname{lin}(H_S \cup \{v\})$ for all $S \in \mathcal{S}$.*
(b) *The splits of a bipyramid $\operatorname{BiPyr} P$ over P with additional vertices v, v' are induced by all hyperplanes $H = \operatorname{lin}(H_S \cup \{v\})$ with $v' \in H$ for all $S \in \mathcal{S}$ and by $\operatorname{lin} P$.*

Proof. First we remark that in all any hyperplane H that induces a split on $\operatorname{Pyr} P$ or $\operatorname{BiPyr} P$ has to meet $\operatorname{relint} P$. Suppose that is not the case. Then v (or v') is on one side of H and P on the other and each edge (v, w) for some vertex w of P is cut in the interior. Now we proceed with the proof of (a) and (b)

(a) Since P is a facet of $\operatorname{Pyr} P$, by Corollary 2.4 and our considerations above, each splitting hyperplane H_S of $\operatorname{Pyr} P$ has to induce a split on P. So, if $v \notin H_S$ we can take a vertex w of P on the opposite side of H_S as v, and (v, w), which is an edge of $\operatorname{Pyr} P$, is cut in the interior. So the given splits are the only possible ones. To see that all those are splits recall that the edges of $\operatorname{Pyr} P$ are the edges of P together with the edges (v, w) for all vertices w of P. Non of these edges can be cut by H_S, the former because H_S defines a split on P, the latter because $v \in H_S$.
(b) Since H has to meet $\operatorname{relint} P$, either $H \cap P = P$, in which case we have $H = \operatorname{lin} P$, or H induces a split on P. So again, there are only the hyperplanes of the form $H = \operatorname{aff}(H_S \cup \{v\})$ or $H = \operatorname{aff}(H_S \cup \{v'\})$ left. But if these are not equal, we can argue as above to find an edge that H meets in its interior.

□

In terms of split polyhedra, this means that $\operatorname{SplitPoly}(P) \cong \operatorname{SplitPoly}(\operatorname{Pyr} P)$.

The join of two polytopes $P_1 \subset \mathbb{R}^{d_1+1}, P_2 \subset \mathbb{R}^{d_2+1}$ is defined as the common convex hull of $\pi_1(P_1)$ and $\pi_2(P_2)$ in $\mathbb{R}^{d_1+d_2+2}$, where $\pi_i : \mathbb{R}^{d_i+1} \to \mathbb{R}^{d_1+d_2+2}$ are embeddings such that the affine spaces $\pi_1(\mathbb{R}^{d_1+1})$ and $\pi_2(\mathbb{R}^{d_2+1})$ are skew, that is, they have empty intersection and are not parallel. The dimension of $P_1 * P_2$ is $d_1 + d_2 + 1$, providing that $\dim P_i = d_i$.

PROPOSITION 3.25. *Let $P = P_1 * P_2$ be the join of two polytopes P_1 and P_2. Then a subdivision of P is a split if and only if it is a join of a split of P_1 and the trivial subdivision of P_2, or it is a join of a split of P_2 and the trivial subdivision of P_1.*

PROOF. By [**19**, Theorem 4.3.6], the join

$$\Sigma_1(P_1) * \Sigma_2(P_2) := \{F_1 * F_2 \mid F_1 \in \Sigma_1(P_1), F_2 \in \Sigma_2(P_2)\}$$

is a subdivision of P, and all subdivisions of P are of this form for some subdivisions of P_1, P_2. However, a subdivision is a split if and only if it has exactly two maximal faces, and for a join $\Sigma * \Sigma'$ of subdivisions this is the case if and only if Σ has two maximal faces and Σ' two, or the other way around. □

For other constructions, for example for prisms and products, it is much harder to say something about the splits. We consider the following example.

EXAMPLE 3.26. Let S be a square and Q another quadrangle, which is not a parallelogram. Obviously, S and Q both have two splits, corresponding two the two pairs of non-adjacent vertices. However, the prism over S has more splits then the prism over Q: The hyperplane spanned by one edge in the upper part and the opposite edge in the lower part is a split of S but not of Q since we assumed that these two edges are not parallel; see also Example 8.36

So there is no chance to get all splits of a product $P_1 \times P_2$ by simply examining all splits of the factors P_1 and P_2. However, we can give at least one sufficient and one necessary condition:

LEMMA 3.27. *Let P_1 and P_2 be two polytopes and $P = P_1 \times P_2 \subseteq \mathbb{R}^{d_1+1} \times \mathbb{R}^{d_2+1}$ their product.*

(a) *If H defines a split of P_1, then $H \times \mathbb{R}^{d_2+1}$ defines a split of P.*
(b) *If H defines a split S of P, then for all vertices (v_1, v_2) of P_S the hyperplane H defines a split or a non-trivial face of $\{v_1\} \times P_2$ and of $P_1 \times \{v_2\}$.*

PROOF. (a) All edges of P are of the form $\{v_1\} \times E_2$ or $E_1 \times \{v_2\}$, where v_1, v_2, E_1, E_2 are vertices and edges of P_1, P_2, respectively. Obviously, no edge of the form $\{v_1\} \times E_2$ can be met by $H \times \mathbb{R}^{d_2+1}$ in the interior, since $\{v_1\} \times E_2$ is parallel to $H \times \mathbb{R}^{d_2+1}$. For the edges of the form $E_1 \times \{v_2\}$ let $E_1 := \operatorname{conv}\{v, v'\}$ and $H = \{x \in \mathbb{R}^{d_1+1} \mid \langle a, x \rangle = 0\}$, so $H \times \mathbb{R}^{d_2+1}$ is given by $\langle (a, 0), y \rangle = 0$. If now $E_1 \times \{v_2\}$ would be met in the interior by $H \times \mathbb{R}^{d_2+1}$, this would mean that $\langle (a, 0), (v, v_2) \rangle > 0$ and $\langle (a, 0), (v', v_2) \rangle < 0$, or vice versa, which is equivalent to $\langle a, v \rangle > 0$ and $\langle a, v' \rangle < 0$. This is a contradiction to the assumption that H defines a split on P_1.
(b) For all vertices (v_1, v_2) of P_S we have that $\{v_1\} \times P_2$ and $P_1 \times \{v_2\}$ are faces of P. So, by Proposition 3.1, the hyperplane H has to define a split or a face of $\{v_1\} \times P_2$ and $P_1 \times \{v_2\}$. Since $(v_1, v_2) \in H$ these faces has to be non-trivial.

□

3.4. Hyperplane Arrangements

Since splits are induced by hyperplanes, given a split system $\mathcal{S}$ one can form an associated hyperplane arrangement. This will be studied in this section. For the theory of hyperplane arrangements see for example the monograph of Orlik and Terao [**72**]. We first recall some basic definitions. A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{R}^d$ is a collection of $(d-1)$-dimensional affine subspaces. It induces a polyhedral subdivision $\Delta_{\mathcal{A}}$ of $\mathbb{R}^d$ (potentially with additional vertices) which is defined as the common refinement of the subdivisions $\Delta_H := \{\{x \in \mathbb{R}^d \mid a \cdot x > b\}, H, \{x \in \mathbb{R}^d \mid a \cdot x < b\}\}$ for all $H \in \mathcal{A}$ where $H = \{x \in \mathbb{R}^d \mid a \cdot x = b\}$. A hyperplane arrangement $\mathcal{A}$ is called *central* if $\bigcap_{H \in \mathcal{A}} \neq \emptyset$.

If we now consider a polytope P and a set $\mathcal{S}$ of splits for P we can form the associated hyperplane arrangement $\mathcal{A}(\mathcal{S}) := \{H_S \mid S \in \mathcal{S}\}$. Note, that in contrast to previous assumptions, in this section, we assume that $P \subset \mathbb{R}^d$ is d-dimensional, hence splits are defined by affine hyperplanes.

The following proposition gives us the connection between split subdivisions of P and the corresponding hyperplane arrangements.

Proposition 3.28. *Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope and $\mathcal{S}$ a weakly compatible split system for P. Then the following holds:*

(a) The tight span $\mathcal{T}_{\mathcal{S}}(P)$ of $\mathcal{S}$ is (isomorphic to) a subcomplex of the dual complex of $\Delta_{\mathcal{A}(\mathcal{S})}$.
(b) If $\mathcal{S}$ is totally incompatible, then $\mathcal{A}_{\mathcal{S}}$ is central, and $\mathcal{T}_{\mathcal{S}}(P)$ is isomorphic to the dual complex of $\Delta_{\mathcal{A}(\mathcal{S})}$.

Proof. (a) Since $\mathcal{S}$ is weakly compatible, the subdivision $\Sigma_{\mathcal{S}}(P)$ is the common refinement of the splits $S \in \mathcal{S}$. So the interior faces of $\Sigma_{\mathcal{S}}(P)$ are the cells $C \cap P$ where C is some cell of $\Delta_{\mathcal{A}(\mathcal{S})}$. Since, obviously, the inclusion relation does not change by cutting with P we get a combinatorial isomorphism between the interior faces of $\Sigma_{\mathcal{S}}(P)$ and such cells C of $\Delta_{\mathcal{A}(\mathcal{S})}$ with $C \cap P \neq \emptyset$. The claim follows by dualizing.

(b) That $\mathcal{S}$ is totally incompatible means that the arrangement $\mathcal{A}(\mathcal{S})$ is central and that $\bigcap_{S \in \mathcal{S}} H_S \subset P$. This also implies that for all non-empty $C \in \Delta_{\mathcal{A}(\mathcal{S})}$ the face $C \cap P$ is non-empty. Together with (a), this shows that $\mathcal{T}_{\mathcal{S}}(P)$ is isomorphic to the entire dual complex of $\Delta_{\mathcal{A}(\mathcal{S})}$.

□

Corollary 3.29. *The tight span of a weakly compatible, totally incompatible split system is a zonotope and hence has a unique maximal face.*

Proof. This follows from the fact that a central hyperplane arrangement is dual to a zonotope. See for example [**97**, Theorem 7.16]. □

Example 3.30. (a) Let P be a pentagon and $\mathcal{S}$ a system of two weakly compatible splits of P (see Figure 3.2). Then the dual complex of $\Delta_{\mathcal{A}(\mathcal{S})}$ is a quadrangle, and the subcomplex corresponding to the tight span of $\mathcal{S}$ consists of two adjacent edges.

(b) Let P be an octahedron ($\cong \Delta(2,4)$) and $\mathcal{S}$ a system of two splits for P. Then the dual complex of $\mathcal{A}(\mathcal{S})$ is again a quadrangle, and, since the split system is totally incompatible, it is equal to the tight span of $\mathcal{S}$.

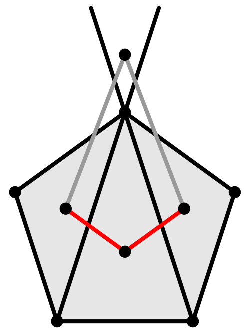

FIGURE 3.2. Two compatible splits of a pentagon, their tight span, and the corresponding hyperplane arrangement.

As an application of Proposition 3.28 and Corollary 3.29, we give a new and much simpler proof of the following statement which was originally given by Huber, Koolen, and Moulton [**51**, Theorem 1.2].

COROLLARY 3.31. *The tight span of a weakly compatible, incompatible split system of* $\Delta(2,n)$ *is a zonotope and hence has a unique maximal face.*

PROOF. By Proposition 8.27, a weakly compatible and incompatible split system of $\Delta(2,n)$ is also totally incompatible. Now use Corollary 3.29. □

Furthermore, we can now give a new proof and a slight strengthening of Theorem 2.35.

PROPOSITION 3.32. *Let* P *be a* d*-dimensional polytope and* $\mathcal{S}$ *a weakly compatible split system for* P*. Then the tight span* $\mathcal{T}_{\mathcal{S}}(P)$ *is a subcomplex of the boundary complex of a* $(d+1)$*-dimensional zonotope and hence a zonotopal complex.*

PROOF. By Proposition 3.28 (a), the tight span of $\mathcal{S}$ is the subcomplex of the dual complex of $\Delta_{\mathcal{A}(\mathcal{S})}$. By homogenizing the hyperplane arrangement $\mathcal{A}(\mathcal{S})$, we get a central hyperplane arrangement $\mathcal{A}'$ in $\mathbb{R}^{d+1}$. The dual complex of $\Delta_{\mathcal{A}'}$ is a zonotope Z, and the dual complex of $\Delta_{\mathcal{A}(\mathcal{S})}$ can be considered as a subcomplex of the dual complex of $\Delta_{\mathcal{A}'}$. So also the tight span $\mathcal{T}_{\mathcal{S}}(P)$ is isomorphic to this subcomplex of the zonotope Z. □

In Question 2.66, we asked which hyperplane arrangements are the arrangements of all split hyperplanes of some polytope. Here we will give an answer to a slightly different question: Which hyperplane arrangements are arrangements of weakly compatible splits systems for some polytope?

THEOREM 3.33. *For each affine hyperplane arrangement* $\mathcal{A}$ *there exists some polytope* P *and a weakly compatible split system* $\mathcal{S}$ *of* P *such that the tight span* $\mathcal{T}_{\mathcal{S}}(P)$ *is isomorphic to the dual complex of* $\Delta_{\mathcal{A}}$.

PROOF. We will assume that $\Delta_{\mathcal{A}}$ does not contain any zero-dimensional cell: If that would be the case, we could just take $\mathcal{A}\times\mathbb{R} := \{H\times\mathbb{R} \mid H\in\mathcal{A}\}$ instead of $\mathcal{A}$, these two arrangement obviously have the same dual complex and $\Delta_{\mathcal{A}\times\mathbb{R}}$ does not contain a zero-dimensional cell. They key observation is the following.

LEMMA 3.34. *Let $P \subseteq \mathbb{R}^d$ be a simplicial, full-dimensional polytope, $\mathcal{S}$ a weakly compatible split system for P and H some hyperplane with $H \cap \operatorname{int} P \neq \emptyset$. If the polyhedral complex $\Delta_{\mathcal{A}}$ corresponding to the hyperplane arrangement $\mathcal{A} := \mathcal{A}(\mathcal{S}) \cup \{H\}$ does not contain a zero-dimensional cell, then there exists some simplicial polytope $\bar{P}$ such that H induces a split S on $\bar{P}$ and $\mathcal{S} \cup \{S\}$ is a weakly compatible split system for $\bar{P}$.*

PROOF. Let us first consider the case that H induces a split S on P. Then the set $\mathcal{S} \cup \{S\}$ is automatically weakly compatible by Corollary 3.7: Since the complex $\Delta_{\mathcal{A}}$ does not contain a zero-dimensional cell, the only edges of $\Sigma_{\mathcal{S}}(P)$ which could be met in the interior are those in the boundary of P. Since P is simplicial all those edges are edges of P, but these cannot be met in the interior since S is a split.

Now we assume that H does not induce a split on P. Then, by Corollary 3.7, there exists some edge E of $\Sigma_{\mathcal{S}}(P)$ such that $\operatorname{relint} E \cap H = \{p\}$ for some $p \in \mathbb{R}^d$. We consider the affine space $A = H \cap \{H \in \mathcal{A} \mid p \in H\}$. By assumption, the space A is at least one-dimensional. We now move p a little bit outside of P to a point $\tilde{p}$ which is still in A and define $\tilde{P} = \operatorname{conv}(P \cup \{\tilde{p}\})$. (Note that this point $\tilde{p}$ is automatically beyond E in the sense of [**97**, Section 3.1], so the combinatorial type of $\tilde{P}$ does not depend on the choice of $\tilde{p}$.)

The new polytope $\tilde{P}$ is still simplicial, and for each facet F of P that contains E, there occur $n-2$ new edges. Non of these edges is cut in the interior by any splitting hyperplane H_S, since this would means that H_S would have cut the simplex F in its interior. We can do this construction again if there would be some other edge E' with $\operatorname{relint} E \cap H \neq \emptyset$ and finally arrive at our desired polytope $\bar{P}$. ◻

We now take an arbitrary hyperplane $H \in \mathcal{A}$ and define P as the union of two (full-dimensional) simplices glued together at some facet contained in H such that additionally for all H' in $\mathcal{A}$ we have $H \cap \operatorname{int} P \neq \emptyset$. This polytope obviously satisfies the hypothesis of Lemma 3.34, and we can piecemeal add all other hyperplanes using Lemma 3.34. ◻

COROLLARY 3.35. *Let $\mathcal{A}$ be a hyperplane arrangement such that $\Delta_{\mathcal{A}}$ does not contain a zero-dimensional cell. Then there exist a polytope P and a weakly compatible splits system $\mathcal{S}$ of P such that $\Delta_{\mathcal{A}} = \Delta_{\mathcal{A}(\mathcal{S})}$.*

3.5. Oriented Matroids

In this section, we will investigate splits of oriented matroids. For the definitions, notations, and results on oriented matroids, we refer to the monograph [**12**] by Björner, Las Vergnas, Sturmfels, White, and Ziegler. We summarize the most important notions.

For sign vectors $X = (X_i), Y = (Y_i) \in \{+,-,0\}^n$, we define $-X, X \circ Y \in \{+,-,0\}^n$ as

$$(-X)_i = -X_i \text{ and } (X \circ Y)_i = \begin{cases} X_i, & \text{if } X_i \neq 0, \\ Y_i, & \text{otherwise.} \end{cases}$$

By $\operatorname{sgn} : \mathbb{R} \to \{+,-,0\}, x \mapsto \begin{cases} +, & \text{if } x > 0, \\ -, & x < 0, \\ 0, & \text{if } x = 0, \end{cases}$ we denote the signum function.

The *support* of a (sign) vector is the set of its nonempty entries.

DEFINITION 3.36. An *oriented matroid* $\mathcal{M}$ on the ground set E is defined by a set $\mathcal{L} \subseteq \{+,-,0\}^{|E|}$ of *covectors* satisfying the following axioms:

(a) $0 \in \mathcal{L}$,
(b) $X \in \mathcal{L}$ implies $-X \in \mathcal{L}$,
(c) $X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$, and
(d) for $X, Y \in \mathcal{L}$ and some i with $X_i = -Y_i \neq 0$ there exist some $Z \in \mathcal{L}$ with $Z_i = 0$ and $Z_j = (X \circ Y)_j = (Y \circ X)_j$ for all j with $X_j = 0$ or $X_j \neq Y_j$.

The *cocircuits* of $\mathcal{M}$ are those covectors with minimal nonempty support. The set of cocircuits is denoted by C. A *face* of $\mathcal{M}$ is a covector $F \in \{+,0\}^{|E|}$, a *facet* of $\mathcal{M}$ is a face which is also a cocircuit, and an *edge* of $\mathcal{M}$ is a face with exactly two positive entries.

The most important examples of oriented matroids are those arising from finite point configurations. The proof of the following proposition is easy and can be found in standard literature about oriented matroids (see e.g., [**12**, Section 1.2 (a)]).

PROPOSITION 3.37. *Let $\mathcal{A} = \{v_1, \dots, v_n\}$ be a set of vectors in $\mathbb{R}^{r+1}$. Then the set*

$$\mathcal{L}(E) : = \left\{(\operatorname{sgn}(\langle a, v_i\rangle))_{1\le i\le n} \;\middle|\; a \in \mathbb{R}^{r+1}\right\}$$

is finite and the set of covectors of an oriented matroid $\mathcal{M}(\mathcal{A})$.

The faces of $\mathcal{M}(\mathcal{A})$ are those $X \in \{+,0\}^{|\mathcal{A}|}$ with $X_i = \begin{cases} + & \text{if } v_i \in F, \\ 0 & \text{otherwise,} \end{cases}$ for some face F of $\operatorname{conv}\mathcal{A}$ and the same holds for the facets and edges.

An oriented matroid which can be received in this fashion is called *realizable.* If we take as $\mathcal{A}$ the set of vertices of an r-dimensional polytope $P \subset \mathbb{R}^{r+1}$, we can define the oriented matroid $\mathcal{M}(P) := \mathcal{M}(\mathcal{A})$ of a polytope P.

Now we are ready to define splits for arbitrary acyclic oriented matroids, motivated by Proposition 3.1.

DEFINITION 3.38. A cocircuit C of an acyclic oriented matroid $\mathcal{M}$ is called a *split* if the restriction of C to each facet F of $\mathcal{M}$ is either a face of F or a split of the restriction of $\mathcal{M}$ to F.

We also get an oriented matroid generalization of some of the equivalences in Proposition 3.1:

PROPOSITION 3.39. *For an oriented matroid $\mathcal{M}$ and a cocircuit C of $\mathcal{M}$ the following are equivalent:*

(a) The restriction of C to all edges E of $\mathcal{M}$ is a face of $\mathcal{M}$ or a split of the restriction of $\mathcal{M}$ to E,
(b) the restriction of C to all faces F of $\mathcal{M}$ is a face of $\mathcal{M}$ or a split of the restriction of $\mathcal{M}$ to F, i.e., C is a split of $\mathcal{M}$,
(c) the restriction of C to all facets F of $\mathcal{M}$ is a face of $\mathcal{M}$ or a split of the restriction of $\mathcal{M}$ to F.

We now compare the notion of a split of a polytope $P \subset \mathbb{R}^{d+1}$ and the corresponding oriented matroid $\mathcal{M}(P)$. Each hyperplane $H = \{x \in \mathbb{R}^{d+1} \mid \langle a, x\rangle = 0\}$ for some $a \in \mathbb{R}^{d+1}$

defines a covector of $\mathcal{M}(P)$ as in Proposition 3.37. One sees that only hyperplanes corresponding to cocircuits can define splits, because the hyperplanes corresponding to non-maximal covectors would meet an edge non-trivially. So, by Proposition 3.1 in connection with Proposition 3.37, we have that a hyperplane H defines a split of P if and only if for the corresponding cocircuit $C(H)$ the restriction of C to each facet F of $\mathcal{M}(P)$ is either a face of F or a split of the restriction of $\mathcal{M}(P)$ to F.

PROPOSITION 3.40. *Let P be a polytope. A hyperplane H induces a split on P if and only if the corresponding cocircuit $C(H)$ defines a split on $\mathcal{M}(P)$.*

COROLLARY 3.41. *Polytopes with the same oriented matroid have the same splits.*

Even more is true: Also the split complex and the weak split complex of a polytope P we defined in Section 2.4 can be obviously generalized to oriented matroids.

PROPOSITION 3.42. *The split complex* $\operatorname{Split}(P)$ *and the weak split complex* $\operatorname{Split}^{\mathrm{w}}(P)$ *of a polytope P only depend on the oriented matroid of P.*

PROOF. Let S_1 and S_2 be two incompatible splits of P. Then there exists a point $x \in \operatorname{relint} P \cap H_{S_1} \cap H_{S_2}$. Here H_{S_i} denotes the split hyperplane of the split S_i. The existence of x is equivalent to the existence of a circuit C in the oriented matroid of P such that C_+ is supported on vertices of P lying on H_{S_1}, that C_- is supported on vertices of P lying on H_{S_2}, and that $C_+ \cup C_-$ is not contained in any facet of P. Since the facets are precisely the positive cocircuits, this is a property of the oriented matroid of P. This shows the claim for the split complex.

The statement for the weak split complex follows from the fact that one can construct common refinements of given subdivisions while only knowing the oriented matroid of the underlying polytope [**19**, Corollary 4.1.43]. □

With the description of splits in Proposition 3.39 (a), one gets a simple algorithm for finding splits of an oriented matroid $\mathcal{M}$: For each cocircuit C of $\mathcal{M}$ check for each edge $\{i, j\}$ of $\mathcal{M}$ if $C_i = C_j$. Then C is a split if and only if this is true for all edges.

If we have an oriented matroid $\mathcal{M}$ with, say, e edges and c cocircuits this naïve algorithm for finding all splits of $\mathcal{M}$ runs in time $O(e \cdot c)$. Of course, by Proposition 3.40, this algorithm can be used also for polytopes. In this case, the algorithm was implemented by the author and is part of the software system `polymake` [**37**].

3.6. Combinatorial Description of Splits

As we have seen in Section 2.6, splits of the second hypersimplex $\Delta(2, n)$ have a very easy combinatorial description as bipartitions of the set $[n]$. The notions of compatibility and weakly compatibility can also be described combinatorially; see Corollary 2.46 and Proposition 2.47. In this section, we try to find similar descriptions for general polytopes.

So let P be a polytope and $H_S = \{x \in \mathbb{R}^{d+1} \mid \langle a, x\rangle = 0\}$ a splitting hyperplane. We define $H^+ = \{x \in \mathbb{R}^{d+1} \mid \langle a, x\rangle \geq 0\}$ and $H^- = \{x \in \mathbb{R}^{d+1} \mid \langle a, x\rangle \leq 0\}$. The combinatorial description of the split S is now given by two disjoint subset L_S, R_S of $\operatorname{Vert} P$ such that

$$L_S = \{v \in \operatorname{Vert} P \mid \langle a, v\rangle > 0\} \quad \text{and} \quad R_S = \{v \in \operatorname{Vert} P \mid \langle a, v\rangle < 0\}.$$

This description is motivated by the oriented matroid point of view (see Section 3.5); the sets L_S and R_S correspond to the elements of $C(H)$ with $C_i = +$ and $C_i = -$, respectively. Of course, one gets the same split by switching L_S and R_S. We denote the set $(\operatorname{Vert} P) \setminus (L_S \cup R_S) = (\operatorname{Vert} P) \cap H_S$ by E_S. By the definition of split, we have that neither L_S, E_S, nor R_S can be empty. We identify S with the set $\{L_S, R_S\}$, and if $A = L_S$ wet $A' := R_S$.

To get a combinatorial description of compatibility we need the following lemma.

LEMMA 3.43. *Let S and T be two splits of a polytope P and $A \in S$, $B \in T$ with $A \subset B$. Then $B' \subset A'$.*

PROOF. We can assume without loss of generality that $A = L_T$ and $B = L_S$, hence $A' = R_T$ and $B' = R_S$. So we have $L_T \subset L_S$, which is equivalent to $L_T \cap (E_S \cup R_S) = \emptyset$. So $R_S \cap L_T = \emptyset$, and since $R_S \subset R_T$ is equivalent to $R_S \cap (E_T \cup L_T) = \emptyset$, it remains to show that $R_S \cap E_T = \emptyset$. Suppose now there exists a vertex v of P with $v \in R_S \cap E_T$. Since $L_T \neq \emptyset$ and $L_T \subset L_S$, there exists another vertex v' of P with $w \in L_T \cap L_S$. Since v and v' are on different sides of H_S, there exists unique point x with $\operatorname{conv}\{v, v'\} \cap H_S = \{x\}$. Obviously, since $v \in H_T^+$ and $v' \in \operatorname{int} H_T^+$, we have that $x \in \operatorname{int} H_T^+$. However, since H_S defines a split of P, by Proposition 3.1 (f), $H_S \cap P = \operatorname{conv}(\operatorname{Vert} P \cap H_S) = \operatorname{conv} E_S$. But since $L_T \cap E_S = \emptyset$, we have $H_S \cap P \subset H_T^-$, contradicting $x \in P \cap H_S \cap \operatorname{int} H_T^+$. □

Note that Lemma 3.43 tells us that in the following proposition the statements in parenthesis are equivalent to those previous to them. The proposition gives a combinatorial condition for two splits to be compatible.

PROPOSITION 3.44. *Two splits S and T of a polytope $P \subset \mathbb{R}^{d+1}$ are compatible if and only if one of the four conditions*

$$\begin{aligned} &L_S \subset L_T \ \ (\textit{or } R_T \subset R_S),\\ &L_S \subset R_T \ \ (\textit{or } L_T \subset R_S),\\ &L_T \subset L_S \ \ (\textit{or } R_S \subset R_T), \quad \textit{or}\\ &R_T \subset L_S \ \ (\textit{or } R_S \subset L_T), \end{aligned}$$

is satisfied or, equivalently, there exists some $A \in \{L_S, R_S\}$ and some $B \in \{L_T, R_T\}$ such that

$$A \subset B \ \ (\textit{or } B' \subset A').$$

PROOF. Suppose first that S and T are compatible. The hyperplanes H_S and H_T partition the space $\mathbb{R}^{d+1}$ into four cones $H_S^+ \cap H_T^+$, $H_S^+ \cap H_T^-$, $H_S^- \cap H_T^+$, and $H_S^- \cap H_T^-$. Since S and T are compatible at least one of these four cones cannot meet the interior of P because otherwise $H_S \cap H_T$ would also. We assume that this is the case for $C := H_S^+ \cap H_T^-$, so we have $C \cap \operatorname{relint} P = \emptyset$ and also $P \cap \operatorname{int} C = \emptyset$. We will now prove that $L_S \subset L_T$.

So suppose that there is some vertex v of P which is in L_S but not in L_T. This means that either $v \in L_S \cap R_T \subset \operatorname{int} C$, which is impossible, or that $v \in L_S \cap E_T$. By Lemma 3.43, we get some vertex v' which is in R_T but not in R_S and it follows similarly that $v' \in R_T \cap E_S$. Consider now some point x in the relative interior of $\operatorname{conv}\{v, v'\}$. Since $v \in L_S$ and $v' \in E_S$, we get that $x \in \operatorname{int} H_S^+$, and since $v \in E_T$ and $v' \in R_T$, we get that $x \in \operatorname{int} H_T^-$. So we have $x \in \operatorname{int} C \cap P$, which is a contradiction.

Now suppose that for two splits S and T we have $L_S \subset L_T$ and $R_T \subset R_S$, and assume that S and T are not compatible. We look at the polytopes $P_S (= H_S \cap P)$ and P_T which has to meet in their relative interior. This means that there exists some vertex v_1 of P_S which is in L_T and some vertex v_2 which is in R_T, since otherwise the interior of P_T would be contained in H_S^+ or H_S^-. So we have $v_1 \in L_T \cap E_S$; hence neither $L_T \subset L_S$ nor $L_T \subset R_S$, and $v_2 \in R_T \cap E_S$; hence neither $R_T \subset L_S$ nor $R_T \subset R_S$. This shows the claim. □

For the weakly compatible case, it is much more complicated to give a simple combinatorial criterion. A first step in this direction can be given by using Corollary 3.13 and Lemma 3.12.

PROPOSITION 3.45. *(a) Let S and T be two splits of a d-dimensional polytope P. Then S and T are not weakly compatible if they are incompatible and $|E_S \cap E_T| < d - 1$.*

(b) Let $\mathcal{S}$ be an incompatible split system for a d-dimensional polytope P. Then $\mathcal{S}$ is not weakly compatible if $\left|\bigcap_{S \in \mathcal{S}} E_S\right| < d + 1 - |\mathcal{S}|$.

In the case of a three-dimensional polytope, we also have the converse of Proposition 3.45 (a).

PROPOSITION 3.46. *Let S and T be splits of a d-dimensional polytope P with $d \leq 3$. Then S and T are weakly compatible if and only if they are compatible or we have $|E_S \cap E_T| \geq d - 1$.*

PROOF. For one- or two-dimensional polytopes there is nothing two show, since one-dimensional polytopes have no splits, and there are no splits systems of two-dimensional polytopes that are weakly compatible and incompatible; see Remark 3.10. So let P be three-dimensional and S and T two weakly compatible splits with $|E_S \cap E_T| \geq d - 1 = 2$. Then the two planes H_S and H_T meet in a line, and $H_S \cap H_T \cap \operatorname{Vert} P$ has at least two points. Since P is convex, these can only be the two endpoints of the line segment $H_S \cap H_T \cap P$. So S and T are weakly compatible. □

This observation can be used to provide an algorithm that computes the split subdivisions of a three-dimensional polytope.

CHAPTER 4

Totally Splittable Polytopes

This chapter is joint work with Michael Joswig [**48**]. It is devoted to the investigation of totally splittable polytopes, that is, those polytopes for which each triangulations is a refinement of splits. As can be expected, the assumption of total splittability restricts the combinatorics of P drastically. We prove that the totally splittable polytopes are the simplices, the polygons, the regular cross polytopes, the prisms over simplices, or joins of these. Interestingly, this classification seems to yield precisely those infinite families of polytopes for which the secondary polytopes are known.

This is how the proof (and thus this chapter) is organized: It will frequently turn out to be convenient to phrase facts in terms of a Gale dual of a polytope. Hence we begin our paper with a short introduction to Gale duality and chamber complexes. The first important step towards the classification is the easy Proposition 4.8 which shows that the neighbors of a vertex of a totally splittable polytope must span an affine hyperplane. Then we observe that whenever P is a prism over a $(d-1)$-simplex or a d-dimensional regular cross polytope there is no place for a point v outside P such that $\operatorname{conv}(P \cup \{v\})$ is totally splittable provided that $d \geq 3$. In this sense, prisms and cross polytopes are *maximally totally splittable*. It is clear that the case of $d = 2$ is quite different; and it is one technical difficulty in the proof to distinguish between polygons and higher dimensional polytopes. The next step is a careful analysis of the Gale dual of a totally splittable polytope which allows to recognize a potential decomposition as a join. And a final reduction argument allows to concentrate on maximally totally splittable factors, which then can be identified again via their Gale duals.

4.1. Splits and Gale Duality

Let $\mathcal{A}$ be a configuration of n (not necessarily distinct) non-zero vectors in $\mathbb{R}^{d+1}$ which linearly spans the whole space. We consider the $n \times (d+1)$-matrix V whose rows are the points in $\mathcal{A}$ and assume that V has full rank. As we have seen in Section 3.5, such a vector configuration gives rise to an oriented matroid. For a linear form $a \in (\mathbb{R}^{d+1})^\star$ we have a covector $C^\star \in \{0, +, -\}^V$ by

$$C^\star(v) \ := \ \begin{cases} 0 & \text{if } av = 0\,, \\ + & \text{if } av > 0\,, \\ - & \text{if } av < 0\,. \end{cases}$$

For $\epsilon \in \{0, +, -\}$ we let $C^\star_\epsilon := \{v \in V \mid C^\star(v) = \epsilon\}$, and we call the multiset $C^\star_0$ the *support* of $C^\star$.

Now consider the $n \times (n-d-1)$-matrix $V^\star$ of full rank $n-d-1$ satisfying $V^\mathsf{T} V^\star = 0$; that is, the columns of $V^\star$ form a basis of the kernel of V^T. Then the configuration of row vectors of $V^\star$ is called a *Gale dual* of V. The Gale dual of V is uniquely determined up to affine equivalence. Each vector $v \in V$ corresponds to a row vector $v^\star$ of $V^\star$, called the *vector dual* to v. Throughout, we will assume that all dual vectors are either zero or have unit Euclidean length. If $v^\star$ is zero, all vectors other than v span a linear hyperplane not containing v. We call V *proper* if $V^\star$ does not contain any zero vectors. For the remainder of this section we will assume that V is proper whence $V^\star$ can be identified with a configuration of n points on the unit sphere $\mathbb{S}^{n-d-2}$. Notice that these n points are not necessarily pairwise distinct, even if the vectors in V are. The connection between Gale duality and oriented matroids is the following: The circuits of V are precisely the cocircuits of $V^\star$ and conversely.

Now let P be a d-dimensional polytope in $\mathbb{R}^d$ with n vertices. By homogenizing the vertices $\operatorname{Vert} P$, we obtain a configuration V_P of n non-zero vectors in $\mathbb{R}^{d+1}$ which positively span the whole space. The cocircuits of V_P are given by the linear hyperplanes spanned by vectors in V_P. The vector configuration V_P is proper if and only if P is not a pyramid, and we will assume that this is the case. The *Gale dual* of P is the spherical point configuration $\operatorname{Gale}(P) := V_P^\star$, which again is unique up to (spherical) affine equivalence. See Figure 4.1 for an example.

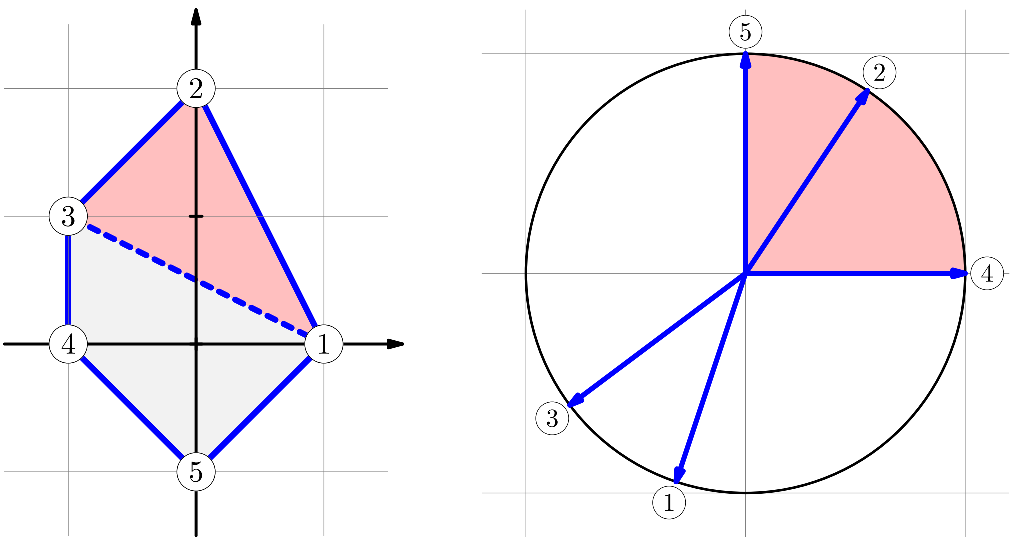

FIGURE 4.1. Pentagon and Gale dual. Corresponding vertices and dual vectors are labeled alike.

EXAMPLE 4.1. The matrices

$$V := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad V^\star := \begin{pmatrix} -1/3 & -1 \\ 2/3 & 1 \\ -4/3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are Gale duals of each other. The rows of the matrix V are the homogenized vertices of the pentagon shown to the left in Figure 4.1. The Gale dual obtained from projecting $V^\star$ to $\mathbb{S}^1$ is shown to the right.

We now intend to study the polytopal subdivisions of our polytope P via Gale duality. Fix a Gale dual $G := \mathrm{Gale}(P)$. Each subset $I \subseteq [n]$ corresponds to a set of (homogenized) vertices V_I. We set $I^\star := [n] \setminus I$ and $V_I^\star := \{v_i^\star \mid i \in I\}$. Then the set V_I affinely spans $\mathbb{R}^d$ if and only if the duals of the complement, that is, the set

$$V_{I^\star}^\star = \{v_i^\star \mid i \in [n] \setminus I\}$$

is linearly independent. In particular, for each d-dimensional simplex $\mathrm{conv}\, V_J$ with $|J| = d+1$ the set $\mathrm{pos}\, V_{J^\star}^\star \cap \mathbb{S}^{n-d-2}$ is a full-dimensional spherical simplex, which is called the *dual simplex* of $\mathrm{conv}\, V_J$. The *chamber complex* $\mathrm{Chamber}(P)$ is the set of subsets of $\mathbb{S}^{n-d-2}$ arising from the intersections of all the dual simplices. The following theorem by Billera, Gel′fand, and Sturmfels [**7**] (see also [**19**, Section 5.3]) is essential.

THEOREM 4.2. [**7**, Theorem 3.1] *The chamber complex* $\mathrm{Chamber}(P)$ *is anti-isomorphic to the boundary complex of the secondary polytope* $\mathrm{SecPoly}(P)$.

The next lemma explains how splits can be recognized in the chamber complex. We continue to use the notation introduced above: in particular, P is the polytope and G its spherical Gale dual.

LEMMA 4.3. *A point* $x \in \mathbb{S}^{n-d-2}$ *defines a split of* P *if and only if there exists a unique circuit* C *in* G *such that* $\mathrm{pos}\, x = \mathrm{pos}\, V_{C_+}^\star \cap \mathrm{pos}\, V_{C_-^\star}^\star$.

PROOF. Consider $x \in \mathbb{S}^{n-d-2}$ such that its chamber is dual to a split S of P. Then the split hyperplane H_S defines a unique cocircuit C of P. Equivalently, C is a circuit of G. Moreover, $\mathrm{pos}\, V_{C_+}^\star$ and $\mathrm{pos}\, V_{C_-^\star}^\star$ correspond to the two maximal cells of S, and $\mathrm{pos}\, x = \mathrm{pos}\, V_{C_+}^\star \cap \mathrm{pos}\, V_{C_-^\star}^\star$.

Conversely, let C be the unique circuit of G such that $\mathrm{pos}\, x = \mathrm{pos}\, V_{C_+}^\star \cap \mathrm{pos}\, V_{C_-^\star}^\star$ for some $x \in \mathbb{S}^{n-d-2}$. Then x is a ray of the chamber complex, and hence it is dual to a coarsest subdivision S of P. Since x is the intersection of exactly two dually maximal cells, the subdivision is a split. □

EXAMPLE 4.4. Let P be the pentagon and G its Gale dual from Example 4.1. Then $C = (0+0--)$ is a cocircuit of P corresponding to the split defined by the line through the vertices v_1 and v_3. Clearly, C is also a circuit of G, with $C_+ = \{2\}$ and $C_- = \{4,5\}$. We have $\mathrm{pos}\, v_2^\star = \mathrm{pos}\, V_{\{2\}}^\star \cap \mathrm{pos}\, V_{\{4,5\}}^\star$, and C is the unique circuit of G yielding $\mathrm{pos}\, v_2^\star$ as the intersection of its positive and its negative cone. The two maximal cells of the split are the quadrangle $\mathrm{conv}\, V_{\{2\}^\star}$ and the triangle $\mathrm{conv}\, V_{\{4,5\}^\star}$; see Figure 4.1.

With each split S of P we associate the unique circuit $C(S)$ of G from Lemma 4.3. If $V_{C(S)_+}^\star$ (or $V_{C(S)_-}^\star$) consists of a single vertex v of P, we call S the vertex split for the vertex v and also write $C(v)$ for $C(S)$. Note that $\mathrm{Vert}\, P \setminus C(v)$ is exactly the set of all vertices of P that are connected to v by an edge.

4.2. Totally Splittable Polytopes

We call a polytope *totally splittable* if all regular triangulations of P are split triangulations. We aim at the following complete characterization.

Theorem 4.5. *A polytope P is totally splittable if and only if it has the same oriented matroid as a simplex, a cross polytope, a polygon, a prism over a simplex, or a (possibly multiple) join of these polytopes.*

By Proposition 3.42 the set of splits and their (weak) compatibility only depends on the oriented matroid of P, and hence the notion "totally splittable" also depends on the oriented matroid only. The *join* $P * Q$ of a d-polytope P and an e-polytope Q is the joint convex hull of P and Q, seen as subpolytopes in mutually skew affine subspaces of $\mathbb{R}^{d+e+1}$. For instance, a 3-simplex is the join of any pair of its disjoint edges. In order to avoid cumbersome notation in the remainder of this section we do not distinguish between any two polytopes sharing the same oriented matroid. For instance, "P is a join of P_1 and P_2" actually means "P has the same oriented matroid as the join of P_1 and P_2" and so on.

Example 4.6. We inspect the classes of polytopes occurring in Theorem 4.5.

(a) Simplices are totally splittable in a trivial way.
(b) A triangulation of an n-gon is equivalent to choosing $n-3$ diagonals which are pairwise non-intersecting. This is a compatible system of splits, and hence each polygon is totally splittable; see Example 2.32. The secondary polytope of an n-gon is the $(n-3)$-dimensional associahedron [**39**, Chapter 7, Section 3.B].
(c) Let $P = \operatorname{conv}\{\pm e_i \,|\, i \in [d]\}$ be a regular cross polytope in dimension d. The splits of P are given by the coordinate hyperplanes $x_i = 0$, for $i \in [d]$. By combining any $d-1$ of these splits one gets a triangulation of P. Conversely, each triangulation of P arises in this way; see Example 2.33. A Gale dual of P is given by the multiset $G \subset \mathbb{S}^{d-2}$ consisting of all points
$$\{e_i \,\big|\, i \in [d-1]\} \cup \left\{ \frac{1}{\sqrt{d-1}} \sum_{i=1}^{d-1} e_i \right\},$$
where each point occurs exactly twice. All the rays in the chamber complex correspond to vertex splits, and the chamber complex is the normal fan of a $(d-1)$-simplex (where each vertex carries two labels). So the secondary polytope of P is a $(d-1)$-simplex. See Figure 4.2 (left) below for $d = 3$.
(d) Let P be the prism over a $(d-1)$-simplex. Then the dual graph of any triangulation of P is a path of with d nodes. The secondary polytope of P is the the $(d-1)$-dimensional permutohedron [**39**, Chapter 7, Section 3.C]. See Figure 4.2 (right) below for $d = 3$.

Remark 4.7. As the secondary polytope of a join of polytopes is the product of their secondary polytopes (e.g., this can be inferred from [**19**, Corollary 4.2.8]), Theorem 4.5 and Example 4.6 show that the secondary polytopes of totally splittable polytopes are (possibly multiple) products of simplices, permutohedra, and associahedra.

It is obvious that total splittability is a severe restriction among polytopes. The following result is a key first step.

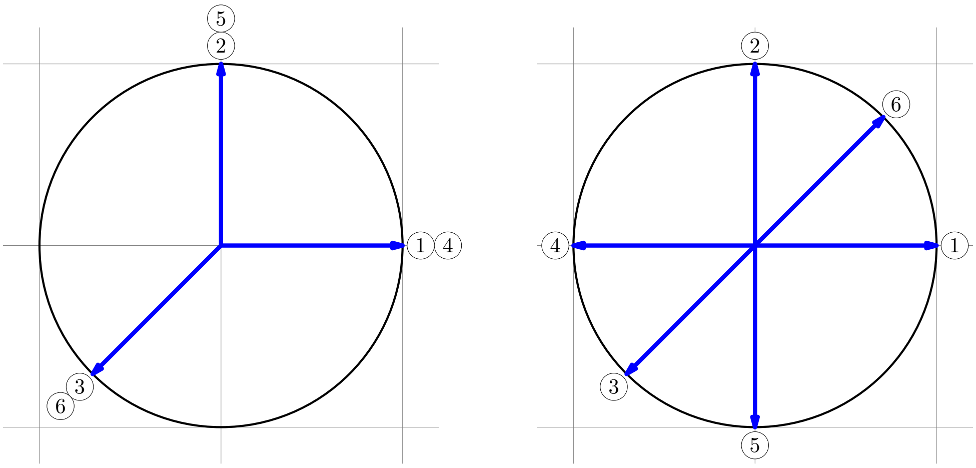

FIGURE 4.2. Gale diagrams of the regular octahedron (left) and of the prism over a triangle (right).

PROPOSITION 4.8. *Let P be a totally splittable polytope. Then each face, each vertex figure, and each subpolytope $Q := \mathrm{conv}(V \setminus \{v\})$ for a vertex $v \in V$ is totally splittable. Moreover, each vertex gives rise to a vertex split, and the neighbors of v span a facet of Q.*

PROOF. Each triangulation of P induces a triangulation on each face F of P. A split of P either does not separate F, or it is a split of F. This implies the total splittability of the faces.

We can assume that P is not a simplex. Let $v \in V$ be a vertex of P. Then there is a placing triangulation Σ of P such that v comes last [**19**, Section 4.3.1]. By assumption, this is a split triangulation, and hence each interior cell of codimension one spans a split hyperplane. Fix a d-simplex $\sigma \in \Sigma$ containing v. Then the facet of σ not containing v is an interior cell of codimension one, and so it spans a split hyperplane H. Since H cannot cut through the other simplices in Σ and v is the vertex placed last to obtain Σ, all neighbors of v in the vertex-edge graph are contained in H. This proves that H is the split hyperplane of the vertex split to v. Moreover, $Q = \mathrm{conv}(V \setminus \{v\})$ is totally splittable, and the vertex figure with respect to v is a facet of Q. This completes the proof. □

REMARK 4.9. The same argument as in the proof above shows: Each hyperplane spanned by d affinely independent vertices of a totally splittable polytope defines a facet or a split.

Note that there exist polytopes for which each vertex defines a vertex split, but which are not totally splittable. An example is the 3-cube which is simple, and hence each vertex defines a vertex split Remark 2.14, but which has several triangulations which are not induced by splits; see Examples 2.19 and 2.34. It is crucial that by Proposition 4.8 the neighbors of a vertex v of a totally splittable polytope span a hyperplane, which we denote by $v^{\perp}$. Proposition 4.8 allows to re-read Lemma 4.3 as follows.

COROLLARY 4.10. *Let v be a vertex of a totally splittable polytope P. Then*

$$v \in \bigcap_{w \in \operatorname{Vert} P \setminus C(v)} w^{\perp} .$$

REMARK 4.11. In the situation of Proposition 4.8 all facets of Q are also facets of P except for the facet F spanning the hyperplane $v^{\perp}$. Moreover, all vertices of Q are also vertices of P. In this situation, we say that v is *almost beyond* the facet F of P. This is slightly more general than requiring that v is *beyond* F, which means that F is the unique facet of P violated by P, and additionally v is not contained in any hyperplane spanned by a facet of P. If v is beyond F and $d = \dim P = \dim Q \geq 3$, then the vertex-edge graph of Q is the subgraph of the vertex-edge graph of P induced on $\operatorname{Vert} P \setminus \{v\} = \operatorname{Vert} Q$. The vertices in the set $\operatorname{Vert} P \setminus C(v)$ mentioned in Corollary 4.10 then are precisely the neighbors of v in the vertex-edge graph of P. In any case, the neighbors of v are precisely the vertices on the facet F of Q.

LEMMA 4.12. *For two polytopes P and Q the join $P * Q$ is totally splittable if and only if both P and Q are.*

PROOF. Suppose that $P * Q$ is totally splittable. Then P and Q both occurs as faces of $P * Q$, and the claim follows from Proposition 4.8.

Let $\dim P = d$ and $\dim Q = e$, and assume that P and Q both are totally splittable. The join of a d-simplex and an e-simplex is a $(d + e + 1)$-simplex, and hence the join cell-by-cell of a triangulation of P and a triangulation of Q yields a triangulation of $P * Q$. Conversely, each triangulation of $P * Q$ arises in this way [**19**, Theorem 4.2.7]. The join of a split hyperplane of P with Q and the join of a split hyperplane of Q with P yields split hyperplanes of $P * Q$. Now consider any triangulation Σ of $P * Q$. Then there are triangulations Σ_P and Σ_Q of P and Q, respectively, such that $\Sigma = \Sigma_P * \Sigma_Q$. By assumption, there is a set S_P of splits of P inducing Σ_P. Likewise S_Q is the set of splits inducing Σ_Q. Then the set of joins of all splits from S_P with $\operatorname{aff} Q$ (as an affine subspace of $\mathbb{R}^{d+e+1}$) and the set of joins of all splits from S_Q with $\operatorname{aff} P$ jointly induce the triangulation Σ. □

Lemma 4.12 together with Example 4.6 completes the proof that all the polytopes listed in Theorem 4.5 are, in fact, totally splittable. The remainder of this section is devoted to proving that there are no others.

PROPOSITION 4.13. *Let $P \subset \mathbb{R}^d$ be a totally splittable d-polytope. Then P is a regular cross polytope if and only if the intersection $\bigcap_{v \in \operatorname{Vert} P} v^{\perp}$ is not empty.*

PROOF. Clearly, the regular cross polytope $P = \operatorname{conv}\{\pm e_i \mid i \in [d]\}$ has the property that the intersection of its split hyperplanes is the origin. Conversely, suppose that P is not a cross polytope. Then there exists a vertex v of P such that at least two vertices u, w are separated from v by the hyperplane $v^{\perp}$. By Proposition 4.8, the split hyperplane $v^{\perp}$ passes through the neighbors of v in the vertex-edge graph of P. Since u is on the same side of $v^{\perp}$ as w it follows that $v^{\perp} \neq w^{\perp}$ and, moreover, $v^{\perp} \cap w^{\perp} \cap \operatorname{int} P = \emptyset$. Now suppose that the intersection of all split hyperplanes contains points in the boundary of P. But since the split hyperplanes do not cut through edges, the intersection must contain at least one vertex $x \in \operatorname{Vert} P$. But this is a contradiction since $x \notin x^{\perp}$. By a similar argument, we can exclude the final possibility that the intersection of all split hyperplanes contains any points outside P. Therefore this intersection is empty, as we wanted to show. □

In a way, cross polytopes (not being quadrangles) are maximally totally splittable.

LEMMA 4.14. *Let $P \subset \mathbb{R}^d$ be a d-dimensional regular cross polytope and $v \in \mathbb{R}^d \setminus P$ is almost beyond the facet F of P. If $d \geq 3$ then* $\operatorname{conv}(P \cup \{v\})$ *is not totally splittable.*

PROOF. We can assume that $P = \operatorname{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$. Suppose that $\operatorname{conv}(P \cup \{v\})$ is totally splittable. Since we assumed $d \geq 3$ each vertex w of P has at least $d+1$ neighbors. At least d of these are still neighbors of w in $\operatorname{conv}(P \cup \{v\})$, so the hyperplane $w^\perp$ with respect to P is the same as $w^\perp$ with respect to $\operatorname{conv}(P \cup \{v\})$. We have that $F^\perp := \bigcap_{w \in \operatorname{Vert} F} w^\perp = \{0\}$, which implies $v \notin F^\perp$, a contradiction to Corollary 4.10. □

The same conclusion as in Lemma 4.14 holds for prisms over simplices as well. See also Figure 4.3 and Example 4.16 below.

LEMMA 4.15. *Let $P \subset \mathbb{R}^d$ be a prism over a $(d-1)$-simplex and $v \in \mathbb{R}^d \setminus P$ a point which is almost beyond a facet F of P. If $d \geq 3$ then* $\operatorname{conv}(P \cup \{v\})$ *is not totally splittable.*

PROOF. Suppose that $\operatorname{conv}(P \cup \{v\})$ is totally splittable. As in the proof of Lemma 4.14 we are aiming at a contradiction to Corollary 4.10. First suppose that v is beyond F, and hence for $w \in \operatorname{Vert} P$ the hyperplanes $w^\perp$ with respect to P and $\operatorname{conv}(P \cup \{v\})$ coincide, since $d \geq 3$; see Remark 4.11.

Up to an affine transformation we can assume that $P = \operatorname{conv}\{e_1, e_2, \ldots, e_d, f_1, f_2, \ldots, f_d\}$ with

$$f_k \;=\; -\sum_{i \neq k} e_i \,.$$

The neighbors of the vertex e_k are $e_1, e_2, \ldots, e_{k-1}, e_{k+1}, \ldots, e_d$ and f_k; symmetrically for the f_k. A direct computation shows that

$$e_1^\perp \cap e_2^\perp \;=\; \operatorname{aff}\{e_3, e_4, \ldots, e_d, c\} \,,$$

where $c = \frac{1}{2d-2}(e_1 + e_2 + \cdots + e_{d-1} + f_1 + f_2 + \cdots + f_{d-1})\}$ is the vertex barycenter of the facet $G := \operatorname{conv}\{e_1, e_2, \ldots, e_{d-1}, f_1, f_2, \ldots, f_{d-1}\}$, which is a prism.

We have to distinguish two cases: the facet F of P violated by v may be a $(d-1)$-simplex or a prism over a $(d-2)$-simplex. If F is a simplex, for instance, $\operatorname{conv}\{e_1, e_2, \ldots, e_d\}$, then we can conclude that the set $\bigcap_{w \in \operatorname{Vert} v^\perp} w^\perp$ is empty. If, however, F is a prism, for instance, $F = G$, we have

$$\bigcap_{w \in \operatorname{Vert} G} w^\perp \;=\; \{c\} \,.$$

In both cases we arrive at the desired contradiction to Corollary 4.10.

Now suppose that v violates F but it is not beyond F, that is, v is contained in the affine hull of some facet F' of P. Let us assume that $d \geq 4$ and that the assertion is true for $d = 3$. Then the polytope $\operatorname{conv}(F' \cup \{v\})$ is totally splittable by Proposition 4.8. Again, F' may be a $(d-1)$-simplex or a prism over a $(d-2)$-simplex. If F' is a $(d-1)$-simplex, it can easily be seen that $\operatorname{conv}(F' \cup \{v\})$ is not totally splittable for $d > 3$ since F' does not have any splits. If F' is a prism over a simplex, we are done by induction.

An easy distinction of the cases, which we omit, allows to prove the result in the base case $d = 3$. See Example 4.16 and Figure 4.3 for one of the cases arising. □

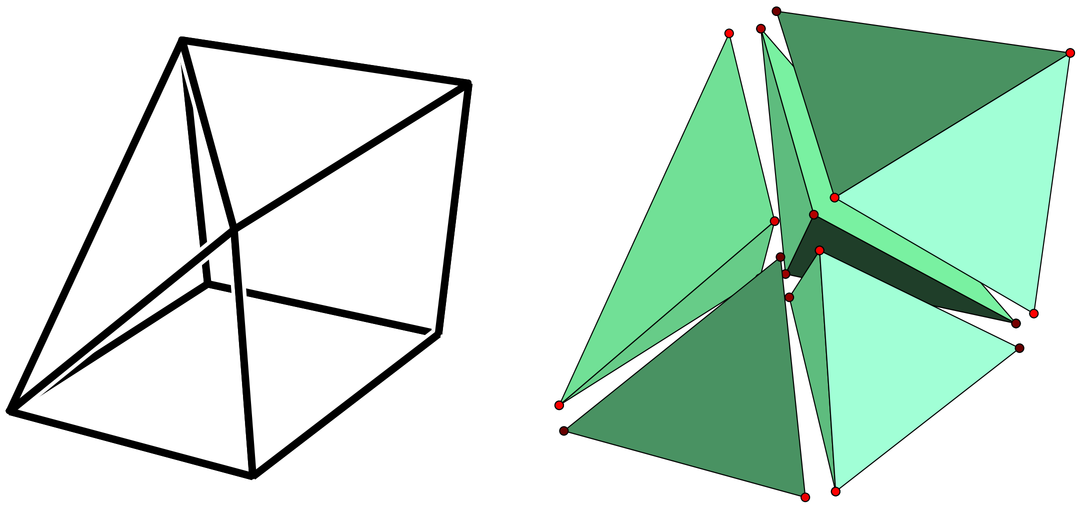

FIGURE 4.3. Convex hull of prism plus one point almost beyond a quadrangular facet, vertex-edge graph (left) and a non-split triangulation (right).

EXAMPLE 4.16. Consider the 3-polytope $P = \text{conv}\{e_1, e_2, e_3, -e_2-e_3, -e_1-e_3, -e_1-e_2\}$, which is a prism over a triangle. For instance, the point $v = e_1+e_2-e_3$ lies almost beyond the quadrangular facet $F = \text{conv}\{e_1, e_2, -e_2 - e_3, -e_1 - e_3\}$. The polytope $\text{conv}(P \cup \{v\})$ admits a triangulation which is not a split triangulation; see Figure 4.3.

PROPOSITION 4.17. *Let P be a proper totally splittable polytope. Then P is a join if and only if the vertex set of P admits a partition $\text{Vert}\, P = U \cup W$ such that no vertex split of a vertex in U is compatible with any vertex split of a vertex in W.*

PROOF. Let $P = (\text{conv}\, U) * (\text{conv}\, W)$ be a proper join. In particular, P is not a pyramid, and $\text{conv}\, U$ and $\text{conv}\, W$ both are at least one-dimensional. Then each vertex in U shares an edge with each vertex in W, and thus the corresponding vertex splits are not compatible.

Conversely, assume that no split with respect to a vertex in U is compatible with a split with respect to any vertex in W. Then each vertex in U shares an edge with each vertex in W. Proposition 4.8 says that each vertex split hyperplane $u^\perp$ contains all neighbors of u. Thus we infer that $\bigcap_{u \in U} u^\perp \supset \text{conv}\, W$ and, symmetrically, we have $\bigcap_{w \in W} w^\perp \supset \text{conv}\, U$.

Now there are two cases to distinguish. Either $\bigcap_{v \in \text{Vert}\, P} v^\perp$ is non-empty. Then P is a regular cross polytope due to Proposition 4.13. But then the vertices of the regular cross polytope do not admit a partition of the required kind, and so this case does not occur. The remaining possibility is that $\bigcap_{v \in \text{Vert}} v^\perp$ is empty. Since all the vertices in U are joined to all the vertices in W, the affine subspaces $\text{aff}\, U$ and $\text{aff}\, W$ must be skew. It follows that $P = (\text{conv}\, U) * (\text{conv}\, W)$. □

For the following we will switch from the primal view on our polytope P to its Gale dual G. A point of multiplicity two in G is called a *double point*.

LEMMA 4.18. *Let P be a totally splittable polytope which is not a join, and let G be a spherical Gale diagram of P. Then P is proper, and each point of G is a single point, or each point is a double point. In particular, there are no points in G with multiplicity greater than two.*

PROOF. Since we assume that P is not a join, in particular, it is not a pyramid, and this is why P is proper. If G had a point with multiplicity three or above this would yield a contradiction to Lemma 4.3.

So suppose now that v_1 is a vertex that has a *sibling* v_2, meaning their duals coincide, and that the set W of all vertices without a sibling is non-empty. Then, again by Lemma 4.3, $v_1^\star = v_2^\star$ is not contained in $\operatorname{pos} W^\star$. By the Separation Theorem, there is an affine hyperplane in $\mathbb{R}^{n-d-1}$ which weakly separates $v_1^\star = v_2^\star$ from $\operatorname{pos} W^\star$. This argument even works for all sibling vertices simultaneously. That is H weakly separates the duals of the sibling vertices from the duals of the non-sibling vertices. By rotating H slightly, if necessary, we can further assume that H contains at least one dual vertex $w^\star$ of a non-sibling vertex $w \in W$. For each such $w \in W$ with $w^\star \in H$ the support of the circuit $C(w)$ is a subset of W^* and from Lemma 4.3 it follows that the support of $C(w)$ is contained in the hyperplane H. In the primal view, this means that all vertices v of P with $v^\star \notin H$ have to be in the splitting hyperplane $w^\perp$ and that the vertex split of w cannot be compatible to any vertex split of a vertex v with $v^\star \notin H$. If now we define $U := \{w \in \operatorname{Vert} P \mid w^\star \in H\}$ we have a partition of $\operatorname{Vert} P$ in U and $\operatorname{Vert} P \setminus U$ such that no vertex split of a vertex in U is compatible with any vertex split of a vertex in $\operatorname{Vert} P \setminus U$. So P is a join by Lemma 4.17. □

LEMMA 4.19. *Let P be a totally splittable d-polytope with $d \geq 2$ which is not a join, and let G be a spherical Gale diagram of P.*

(a) If each point in G is a double point then P is a regular cross polytope.
(b) If each point in G is antipodal then P is a prism over a simplex.
(c) If each point in G is both a double point and antipodal, then $d = 2$, and P is a quadrangle.

A point $x \in G$ is *antipodal* if $-x$ is also in G. Notice that any quadrangle, regular or not, has a zero-dimensional spherical Gale diagram with exactly two pairs of antipodal points.

PROOF. First note that (c) follows from (a) and (b) since $d = 2$ is the only case in which the regular cross polytope is also a prism over a simplex. Suppose that each point in G is antipodal. Then we claim that the number of vertices of P equals $n = 2d$. Let $k := n - d - 1$ be the dimension of the linear span of G and assume for now that $k \geq 2$. Then we can pick $k-1$ pairwise distinct points in G which span an affine hyperplane H. Since G positively spans $\mathbb{R}^k$, there must be at least one point in G on either side of H. This shows that $G \subset \mathbb{S}^{k-1}$ contains at least $k + 1 = n - d$ pairwise distinct points. Now suppose that we have at least $k + 2$ pairs. Take any point $v^\star$ in G and pick an affine hyperplane $H^\star$ orthogonal to $v^\star$ such that $H^\star$ does not contain the origin. Since all $g \in G$ are antipodal, for at least $k + 1$ dual vectors in G the corresponding rays intersect H^*. Without loss of generality we can assume that these $k+1$ dual vectors linearly span $\mathbb{R}^k$. The $k+1$ points of intersection in $H^\star$ span a $(k-1)$-polytope with $k+1$ vertices. Such a

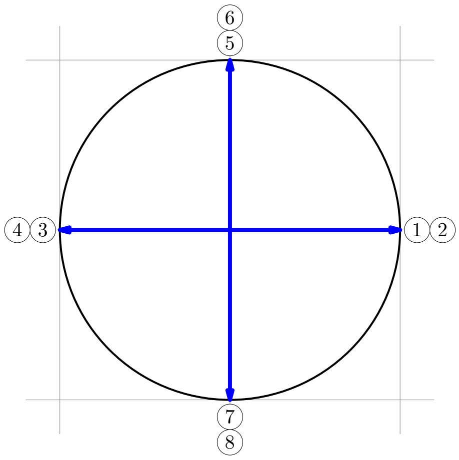

FIGURE 4.4. Gale diagram of the join of two squares, labeled $\{1,2,3,4\}$ and $\{5,6,7,8\}$, respectively.

polytope admits two distinct triangulations both of which yield a minimal circuit whose support contains $v^\star$. This contradicts Lemma 4.3.

Now consider the case $k = 1$. Then, by Lemma 4.18, either $n = 2$ or $n = 4$. If $n = 2$ then $d = 0$, and this case does not occur. If $n = 4$ then $d = 2$, and P is a quadrangle.

For all $k \geq 1$ we proved that G consists of precisely $k+1$ antipodal pairs in $\mathbb{S}^{k-1}$, that is $n = 2k+2$. Up to affine equivalence we can assume that the first k pairs are $\pm e_i$. Since $e_1, e_2, \ldots, e_k$ are not positively dependent, one of the two remaining points $\pm x$ must be contained in the non-negative orthant ${\mathbb{R}_{\geq 0}}^k$. Assume that x is in the boundary of ${\mathbb{R}_{\geq 0}}^k$. Without loss of generality x (and thus also $-x$) is contained in the hyperplane $x_1 = 0$. But then the open halfspaces $x_1 > 0$ and $x_1 < 0$ contain only one point each, namely $\pm e_1$. This is impossible for the Gale dual of a polytope; the argument is the same as above where we excluded a Gale diagram in $\mathbb{S}^0$ consisting of two antipodal points. We conclude that x is in the strictly positive orthant. It can be shown that P is a prism over a k-simplex.

Now assume that each point in G is a double point. Let v be any vertex of P and $v^\perp$ the hyperplane corresponding to the vertex split of v. Since $v^\star$ is a double point in G there is exactly one vertex w other than v which is not contained in $v^\perp$. The polytope $Q := \operatorname{conv}(\operatorname{Vert} P \setminus \{v, w\}) = P \cap v^\perp$ is a face of the vertex figure of v and hence totally splittable by Proposition 4.8. Clearly, a spherical Gale diagram of Q again has only double points. Inductively, we can thus assume that Q is a regular cross polytope. Therefore, its split hyperplanes have a non-empty intersection. Since this intersection is contained in $v^\perp$ it follows that the split hyperplanes of P also have a non-empty intersection. Hence P is a regular cross polytope by Proposition 4.13. As a basis of the induction we can consider the case where G is contained in $\mathbb{S}^1$. As G must span $\mathbb{R}^2$, and as each point in G occurs twice, the polytope P has six vertices, and it is three-dimensional. It can be shown that P is a regular octahedron. □

Now we have all ingredients to prove the main result of this chapter.

Proof of Theorem 4.5. Let P be a totally splittable d-polytope with spherical Gale dual G. By Lemma 4.12, we can assume without loss of generality that P is not a join. Consider a vertex $v \in \operatorname{Vert} P$ with the property that $v^\star$ is neither a double nor an antipodal point. By Proposition 4.8, the polytope $Q := \operatorname{conv}(\operatorname{Vert} P \setminus \{v\})$ obtained from P by the deletion of v is again totally splittable. Moreover, $\dim Q = d$ since P is not a pyramid.

Let us assume for the moment that Q is also not a join. Then we can repeat this procedure until after finitely many steps we arrive at a polytope P' with a spherical Gale diagram G' which consists only of double and antipodal points. In this situation Lemma 4.18 implies that all points of G' are double points or all points of G' are antipodal. By Lemma 4.19, we can conclude that either $d = \dim P = \dim P' = 2$ and P' is a quadrangle, or $d \geq 3$ and P' is a regular cross polytope, or $d \geq 3$ and P' is a prism over a simplex. The question remaining is whether P and P' can actually be different. For $d \geq 3$ this is ruled out by Lemma 4.14 (if P' is a cross polytope) and Lemma 4.15 (if P' is a prism). In the final case $\dim P = \dim Q = \dim P' = 2$.

The proof of our main result will be concluded with the subsequent proposition. □

Proposition 4.20. *Let P be a totally splittable polytope with spherical Gale diagram G, and let v be a vertex of P with the property that its dual $v^\star$ in G is neither a double nor an antipodal point. If P is not a join then $Q := \operatorname{conv}(\operatorname{Vert} P \setminus \{v\})$ is neither.*

Proof. By [**7**, Lemma 3.4], the Gale transform of Q is the minor $G/v^\star$ obtained by contracting $v^\star$ in G. Up to an affine transformation we can assume that $v^\star$ is the first unit vector in $\mathbb{R}^{n-d-1}$, and so $G/v^\star$ is the projection of $G \setminus \{v^\star\}$ to the last $n-d-2$ coordinates. We call the projection map π. Since $v^\star$ is neither antipodal nor a double point, no point in $G/v^\star$ is a loop, and thus Q is proper, that is, it is not a pyramid.

So suppose that $Q = Q_1 * Q_2$ is a join with $\dim Q_1 \geq 1$ and $\dim Q_2 \geq 1$. Then there are spherical Gale diagrams G_1 and G_2 of Q_1 and Q_2, respectively, such that $G/v^\star = G_1 \sqcup G_2$ as a multiset in $\mathbb{S}^{n-d-3}$. Up to exchanging the roles of Q_1 and Q_2, there is a facet F_1 of Q_1 such that the facet $v^\perp$ opposite to v is a join $F_1 * Q_2$. That is to say, the support of the circuit $C(v)$, corresponding to the vertex split of v in P, is mapped to G_1 by π. In particular, $v^\star$ is not in the positive hull of the points dual to the vertices of Q_2. The Separation Theorem implies that there is a linear hyperplane H in $\mathbb{R}^{n-d-1}$ separating $v^\star$ from the duals of the vertices of Q_2. As in the proof of Lemma 4.18 we can now argue that P is a join, which contradicts our assumptions. □

This finally completes the proof of the theorem.

Remark 4.21. If $v^\star$ is antipodal or a double point, then Q is a pyramid over the unique facet of Q which is not a facet of P. This shows that the assumption on $v^\star$ in Proposition 4.20 is necessary. For instance, by inspecting the two Gale diagrams in Figure 4.2 one can see directly that if P is a regular octahedron or a prism over a triangle, in both cases Q is a pyramid over a quadrangle.

We are indebted to Raman Sanyal for sharing the following observation with us.

Corollary 4.22. *Each totally splittable polytope is equidecomposable.*

A polytope is *equidecomposable* if each triangulation has the same f-vector.

Proof. This follows from the classification case by case: Each triangulation of an n-gon has exactly $n-2$ triangles. Each triangulation of a d-dimensional regular cross polytope has exactly $2d-2$ maximal cells. Each triangulation of a prism over a $(d-1)$-simplex has exactly d maximal cells. Observe that equidecomposability is preserved under taking joins. □

It would be interesting to know if Corollary 4.22 has a direct proof without relying on Theorem 4.5.

CHAPTER 5

Splits and Tight Spans of Point Configurations

In the previous chapters, we discussed splits and split subdivisions of convex polytopes. In this chapter, we will regard a slightly more general situation. Instead of only looking at convex polytopes, that is to say point configuration in convex position, we will consider arbitrary point configurations, that is (finite) multisets $\mathcal{A} \subset \mathbb{R}^{d+1}$.

The definition of subdivision for a point configuration $\mathcal{A}$ is not as obvious as for polytopes. One the one hand, one could simply take a geometric point of view and define a subdivision of $\mathcal{A}$ as a polytopal decomposition Σ of $\operatorname{conv} \mathcal{A}$ such that all vertices (i.e., zero-dimensional faces) of Σ are elements of $\mathcal{A}$. We will call such a subdivision a *geometric subdivision* of $\mathcal{A}$. On the other hand, this geometric definition does not take into account the whole structure of the point set (e.g., multiple points in $\mathcal{A}$ does not have any meaning for geometric subdivisions) and does not allow a theory of secondary polytopes. Therefore, we call a set of sub-point configurations (i.e., sub-multisets) of $\mathcal{A}$ an *abstract subdivision* Δ of $\mathcal{A}$ if the following three conditions hold (see [**19**, Section 2.3]):

(a) If $F \in \Delta$ and $\bar{F}$ is a face of F, then $\bar{F} \in \Delta$,
(b) $\operatorname{conv} \mathcal{A} = \bigcup_{F \in \Delta} \operatorname{conv} F$,
(c) If $F, \bar{F} \in \Delta$, then $\operatorname{relint}(\operatorname{conv} F) \cap \operatorname{relint}(\operatorname{conv} \bar{F}) = \emptyset$.

As in the case of polytopes, a point configuration $F \subset \mathcal{A}$ is called a *face* of $\mathcal{A}$ if there exists a supporting hyperplane H of $\operatorname{conv} \mathcal{A}$ such that $F = \mathcal{A} \cap H$.

If we have an (abstract) subdivision Δ of a point configuration $\mathcal{A}$, we can look at the corresponding geometric subdivision $\Sigma(\Delta) := \{\operatorname{conv} F \mid F \in \Delta\}$. On the other hand, for a geometric subdivision Σ of $\mathcal{A}$ there generally does not have to exist a unique abstract subdivision $\Delta(\Sigma)$ with $\Sigma(\Delta(\Sigma)) = \Sigma$. However, we can define $\Delta_{\mathcal{A}}(\Sigma) := \{F \cap \mathcal{A} \mid F \in \Sigma\}$ as the coarsest possible subdivision with that property. Note that if $\operatorname{conv} \mathcal{A} = \operatorname{conv} \mathcal{A}'$ we can have geometric subdivisions Σ, Σ' of $\mathcal{A}$, $\mathcal{A}'$, respectively, with $\Sigma = \Sigma'$ but $\Delta_{\mathcal{A}}(\Sigma) \neq \Delta_{\mathcal{A}'}(\Sigma)$.

Most of the results we proved about tight spans and splits of polytopes remain valid for point configurations. However, in this more general context, we do not only have to deal with splits but also with 1-splits of point configurations. A 1-*split* of a point configuration $\mathcal{A}$ is a subdivision of $\mathcal{A}$ with exactly one maximal face containing all but one of the points in $\mathcal{A}$. It will turn out that these one-splits are needed to prove a generalization of the Split Decomposition Theorem for point configurations.

This chapter is organized as follows. First we will extend the notions of tight span and coherency of weight functions from polytopes to general point configurations. The discussion focuses on the differences and specialties compared to the polytope case in Section 2.2. In Section 5.2, we study which polytopal complexes may occur as the tight span of some point configuration or polytope and prove the following two results: Each tight span of a point configuration also occurs as the tight span of some polytope; and for each polytope P there exists a tight span (of another polytope) whose sole maximal cell is P.

The theory of splits and 1-splits for point configurations is developed in Section 5.3 together with a proof of the generalization of the Split Decomposition Theorem. In the last section we discuss totally splittable point configurations, the analogue of the totally splittable polytopes of Chapter 4 and give a partial classification result.

5.1. Coherency of Weight Functions

As in Section 2.2, given a point configuration $\mathcal{A}$ as an $n \times (d+1)$-matrix V and a weight function $w : \mathcal{A} \to \mathbb{R}$, one can define the *envelope* of $\mathcal{A}$ with respect to w as

$$\mathcal{E}_w(\mathcal{A}) \ := \ \left\{x \in \mathbb{R}^{d+1} \ \middle|\ Vx \geq -w\right\}$$

and the *tight span* $\mathcal{T}_w(\mathcal{A})$ of $\mathcal{A}$ as the complex of bounded faces of $\mathcal{E}_w(\mathcal{A})$. We again make the technical assumption that $\mathbb{1}$ is contained in the column span of V. Furthermore, one can define the polyhedron $\mathcal{L}_w(\mathcal{A})$ and prove a duality theorem similar to Proposition 2.3. A *regular geometric subdivisions* $\Sigma_w(\mathcal{A})$ of $\mathcal{A}$ is obtained by projecting the complex of bounded faces of $\mathcal{L}_w(\mathcal{A})$ to $\operatorname{conv} \mathcal{A}$. A *regular abstract subdivision* $\Delta_w(\mathcal{A})$ of $\mathcal{A}$ is obtained by taking all sets $\{p \in \mathcal{A} \,|\, p + w(p)e_{d+1} \in F\}$ for all lower faces F of $\mathcal{L}_w(\mathcal{A})$. From this, one derives that for two lifting function w_1, w_2 we have that $\mathcal{T}_{w_1}(\mathcal{A}) = \mathcal{T}_{w_2}(\mathcal{A})$ implies $\Sigma_{w_1}(\mathcal{A}) = \Sigma_{w_2}(\mathcal{A})$ but not necessarily $\Delta_{w_1}(\mathcal{A}) = \Delta_{w_2}(\mathcal{A})$; see Example 5.1.

It turns out that almost everything we proved in Section 2.2 about coherency is also true for general point configurations. To be more precise, the statements of Lemma 2.2 up to Corollary 2.8 are true literally if one replaces P and $\operatorname{Vert} P$ by $\mathcal{A}$, and this does not depend on whether one considers geometric or abstract subdivisions. In contrast, Corollary 2.9 is only true for abstract subdivisions $\Delta_w(\mathcal{A}), \Delta_{w'}(\mathcal{A})$. However, one has to be careful at some points. We start out with an example.

EXAMPLE 5.1. Consider the point configuration $\mathcal{H}$ consisting of the vertices of the hexagon H from Example 2.1 together with the additional point $(1, 1, 1)$, that is, the columns of the matrix

$$V'^{\mathsf{T}} \ = \ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 1 \end{pmatrix},$$

and the weight functions $w = (0, 0, 1, 1, 0, 0, 0)$ and $\bar{w} = (0, 0, 1, 1, 0, 0, 1)$. A direct computation shows that

$$\mathcal{T}_w(\mathcal{H}) \ = \ \mathcal{T}_{\bar{w}}(\mathcal{H}) \ = \ \mathcal{T}_{w_1}(H) \ = \ [0, (1, -1, 0)] \, .$$

The geometric subdivisions $\Sigma_w(\mathcal{H})$ and $\Sigma_{\bar{w}}(\mathcal{H})$ agree and are equal to the the split subdivision $\Sigma_{w_1}(H)$. However, the abstract subdivisions $\Delta_w(\mathcal{H})$ and $\Delta_{\bar{w}}(\mathcal{H})$ are not equal: The former has the maximal faces $\{1, 2, 3, 5, 7\}$ and $\{3, 4, 5, 6, 7\}$, but the latter the maximal faces $\{1, 2, 3, 5\}$ and $\{3, 4, 5, 6\}$. (Where the numbers correspond to the rows of the matrix V'.) So $\Delta_{\bar{w}}(\mathcal{H})$ is strictly finer then $\Delta_w(\mathcal{H})$. This agrees with the coherency indices (defined as in Equation 2.2) $\alpha^{w}_{\bar{w}} = 0$ and $\alpha^{\bar{w}}_{w} = 1$ and gives us a counterexample to Corollary 2.9 for the case of geometric subdivisions.

The decomposition $(w, \bar{w})$ is coherent and $\Delta_{w+\bar{w}}(\mathcal{H}) = \Delta_{\bar{w}}(\mathcal{H})$, $\Sigma_{w+\bar{w}}(\mathcal{H}) = \Sigma_{\bar{w}}(\mathcal{H})$ in accordance with Corollary 2.4.

If one looks carefully at the proofs in Section 2.2, one sees that the proofs of Lemma 2.2, Proposition 2.3, and Proposition 2.7 also work for point configurations (if one replaces P and $\operatorname{Vert} P$ by $\mathcal{A}$). A bit more care is needed for Corollary 2.4. At first sight, the statement seams to work only for geometric subdivisions (with the same proof) since coherency is defined in terms of the tight span which does not distinguish between different abstract subdivisions which define the same geometric subdivisions. However, if we define $F_w(x)$ simply as $\{v \in \mathcal{A} \mid \langle v, x\rangle = -w\}$ in the proof of Corollary 2.4, the proof works well. The key point is that this definition does not only take into account the tight span $\mathcal{T}_w(\mathcal{A})$ or the envelope $\mathcal{E}_w(\mathcal{A})$, but the vertex-inequality incidence structure of $\mathcal{E}_w(\mathcal{A})$. Indeed, together with its vertex-inequality incidence matrix, the tight span can distinguish between different abstract subdivisions.

Although Corollary 2.4 is true for geometric and abstract subdivisions in the version we stated it, the slightly stronger statement, Corollary 5.2 below, which was implicitly used in the proof of Corollary 2.9, does only hold for abstract subdivisions.

Corollary 5.2 (Strengthening of Corollary 2.4). *A decomposition $w = w_1 + w_2$ of weight functions of $\mathcal{A}$ is coherent if and only if the subdivisions $\Delta_{w_1}(\mathcal{A})$ and $\Delta_{w_2}(\mathcal{A})$ have a common refinement.*

Corollary 5.2 directly follows from the secondary fan theory we also discussed in Section 2.2. This theory works fine for abstract subdivisions, but not for geometric subdivisions; see [**19**, Chapter 5].

Example 5.3. We consider the point configuration $\mathcal{A}$ whose elements are the columns of the matrix

$$V^{\mathsf{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$$

consisting of the vertices of a square together with its center (see Figure 5.1) and the weight function $w_1 = (1,0,0,0,0), w_2 = (0,1,0,0,0), \bar{w_1} = (1,0,0,0,1)$, and $\bar{w_2} = (0,1,0,0,1)$. A computation of the tight spans gives

$$\mathcal{T}_{w_1}(\mathcal{A}) = \mathcal{T}_{\bar{w}_1}(\mathcal{A}) = [0,(-1,1/2,1/2)], \quad \text{and} \quad \mathcal{T}_{w_2}(\mathcal{A}) = \mathcal{T}_{\bar{w}_2}(\mathcal{A}) = [0,(0,-1/2,-1/2)].$$

As in Example 5.1, we have that $\Sigma_{w_i}(\mathcal{A}) = \Sigma_{\bar{w}_i}(\mathcal{A})$, but $\Delta_{w_i}(\mathcal{A}) \neq \Delta_{\bar{w}_i}(\mathcal{A})$ for $i = 1,2$. Hence the geometric subdivisions $\Sigma_{\bar{w}_1}(\mathcal{A})$ and $\Sigma_{\bar{w}_2}(\mathcal{A})$ have a common refinement, the subdivision depicted in Figure 5.1 on the right, just as $\Sigma_{w_1}(\mathcal{A})$ and $\Sigma_{w_2}(\mathcal{A})$. The corresponding abstract subdivision is also the common refinement of $\Delta_{w_1}(\mathcal{A})$ and $\Delta_{w_2}(\mathcal{A})$; but $\Delta_{\bar{w}_1}(\mathcal{A})$ and $\Delta_{\bar{w}_2}(\mathcal{A})$ does not have a common refinement. This agrees with the fact that (w_1, w_2) is coherent whereas $(\bar{w}_1, \bar{w}_2)$ is not and verifies our strengthening of Corollary 2.4 in this case of abstract subdivisions (and falsifies it for geometric subdivisions).

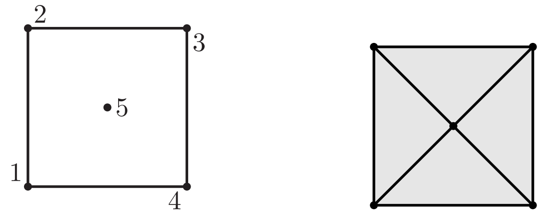

FIGURE 5.1. The point configuration of Example 5.3.

REMARK 5.4. In Remark 2.6, we mentioned the special case of the second hypersimplex $\Delta(2, n)$. One can also consider the point configuration $\mathcal{A}$ consisting of the vertices of $\Delta(2, n)$ together with the n additional points $2e_i$. The convex hull $\mathcal{A}$ is a $(d-1)$-dimensional simplex. Lifting functions $w : \mathcal{A} \to \mathbb{R}_{\geq 0}$ where $w(2e_i) = 0$ for all i, does not correspond to metrics, but to *distance functions* which does not necessarily fulfill the triangle inequality. These were studied by Hirai [**50**, Section 4].

A further special case we mentioned in Remark 2.6 is the Delone decomposition. This is particular interesting also for point configurations, and everything we mentioned in that remark is also true in this generality. In this case, one is usually interested in the geometric and not in the abstract subdivision.

5.2. Tight Spans of Point Configurations and Polytopes

When considering tight spans, one might wonder which polytopal complexes might arise as the tight span of some regular subdivision of a polytope (or a point configuration). In this section, we will give a partial answer to this question: If the tight span is a single polytope, it can be any polytope. It turns out that the generalization of tight spans to point configuration in this chapter is very useful, even if we intrinsically only want to talk about tight spans of polytopes.

To this end, we will first relate tight spans of polytopes and tight spans of point configurations.

PROPOSITION 5.5. *Let $\mathcal{A} \subset \mathbb{R}^{d+1}$ be a point configuration and $\Sigma_w(\mathcal{A})$ a regular subdivision of $\mathcal{A}$. Then there exists a polytope $P \subset \mathbb{R}^{d+2}$ together with a regular subdivision $\Sigma_{w'}(P)$ of P such that $\mathcal{T}_{w'}(P)$ is affinely isomorphic to $\mathcal{T}_w(\mathcal{A})$.*

PROOF. We can assume without loss of generality that $\mathcal{A}$ does not have any multiple points and that for all cells $C \in \Sigma_w(\mathcal{A})$ we have $\mathrm{Vert}(C) = C \cap \mathcal{A}$. Furthermore, we assume that $w < 0$. Then we define the polytope $P \subset \mathbb{R}^{d+2} = \mathbb{R}^{d+1} \times \mathbb{R}$ as

$$P = \mathrm{conv}\,\{(a, \pm w(a)) \mid a \in \mathcal{A}\}\ .$$

From our assumption that every $p \in \mathcal{A}$ is the vertex of some $F \in \Sigma_w(\mathcal{A})$, it follows that all the lifted points $(a, w(a))$ are vertices of $\mathcal{L}_w(\mathcal{A})$; and so from $w < 0$ it follows that all points $(a, \pm w(a))$ are vertices of P. We define our weight function $w' : \mathrm{Vert}\, P \to \mathbb{R}$ as $w'(a, \pm w(a)) = w(a)$. From the definition of the envelope, we directly get that $x \in \mathcal{E}_w(\mathcal{A})$ implies $(x, 0) \in \mathcal{E}_{w'}(P)$ and that $(x, x') \in \mathcal{E}_{w'}(P)$ implies $x \in \mathcal{E}_w(\mathcal{A})$. We will now show that $\mathcal{T}_{w'}(P) = \mathcal{T}_w(\mathcal{A}) \times \{0\}$, which yields the theorem.

Stated otherwise, we have to show that $(v, v') \in \mathbb{R}^{d+1} \times \mathbb{R}$ is a vertex of $\mathcal{E}_{w'}(P)$ if and only if $v' = 0$ and v is a vertex of $\mathcal{E}_w(\mathcal{A})$. So let first v be a vertex of $\mathcal{E}_w(\mathcal{A})$. Then there exists a $(d+1)$-element set $\mathcal{B} \subset \mathcal{A}$ such that v is the unique solution x of the linear system $\langle a, x\rangle = -w(a)$ for all $a \in \mathcal{B}$. This implies that $(v, 0)$ is the unique solution (x, x') to the system $\langle a, x\rangle \pm w(a)x' = -w(a)$ for all $a \in \mathcal{B}$, and so $(v, 0)$ is a vertex of $\mathcal{E}_{w'}(P)$.

On the other hand, consider a vertex $(v, v') \in \mathbb{R}^{d+1} \times (\mathbb{R} \setminus \{0\})$ of

$$\mathcal{E}_{w'}(P) = \left\{(x, x') \in \mathbb{R}^{d+1} \times \mathbb{R} \;\middle|\; \langle a, x\rangle \pm w(a) \geq w(a)x' \text{ for all } a \in \mathcal{A}\right\}.$$

Suppose that there exists some $p, q \in \mathcal{A}$ with

$$\langle p, v\rangle + w(p)v' = -w(p) \quad \text{and} \tag{5.1}$$

$$\langle q, v\rangle - w(q)v' = -w(q)\,. \tag{5.2}$$

Since $v \in \mathcal{E}_w(\mathcal{A})$ we have $\langle p, v\rangle \geq -w(p)$ and $\langle q, v\rangle \geq -w(q)$. Furthermore, by our assumption, we have $w(p), w(q) < 0$. So Equation (5.1) yields $v' \geq 0$, and Equation (5.2) yields $v' \leq 0$, a contradiction. So we can assume that we only have equality in "+"-inequalities. Hence, we find a $(d+2)$-element set $\mathcal{B} \subset \mathcal{A}$ such that (v, v') is the unique solution (x, x') of the linear system $\langle a, x\rangle + w(a)x' = -w(a)$ for all $a \in \mathcal{B}$. However, a solution to this system is $(0, -1)$ which is not an element of $\mathcal{E}_{w'}(P)$ (since it does not fulfill any of the "−"-inequalities). This contradiction finishes the proof. □

REMARK 5.6. One can show that

$$\mathcal{E}_{w'}(P) \;=\; \left\{(x, x') \in \mathbb{R}^{d+1} \times \mathbb{R} \;\middle|\; x \in \mathcal{E}_w(\mathcal{A}) \text{ and } |x'| \leq -\max_{a\in\mathcal{A}} \frac{\langle a, x\rangle}{w(a)} - 1\right\}.$$

By Proposition 5.5, we get that the generalization to point configurations does not yield new tight spans, all possible tight spans still occur as tight spans of polytopes. However, for a point configuration we do not generally find a polytope with the same secondary polytope (or set of subdivisions), so if one is interested in the global structure, point configurations still have to be considered.

Furthermore, Proposition 5.5 gives us the possibility to give examples of d-dimensional point configurations with tight spans that equal tight spans of $(d+1)$-dimensional polytopes. This will be used in Chapter 6. Note however, that if we have a tight span of a triangulation of a point configuration, the tight span of the polytope constructed in the proof of Proposition 5.5 does not again belong to a triangulation. On the other hand, we will see in Section 6.1 that a similar statement is true for coarsest subdivisions.

Now we can prove the main result of this chapter.

THEOREM 5.7. *Let P be any polytope. Then there exists a polytope P' and a lifting function w for P' such that $\mathcal{T}_w(P')$ is affinely isomorphic to P.*

For the proof we need some notions about polytope polarity (very similar to cone polarity we used in the proof of Proposition 2.3). We only give the notions and results we use here and refer the reader to [**97**, Section 2.3] or [**42**, Section 3.4] for details.

For a set $A \subset \mathbb{R}^d$ the *polar set* A° is defined as

$$A^\circ \;=\; \left\{y \in \mathbb{R}^d \;\middle|\; \langle x, y\rangle \leq 1\right\}.$$

If A is a compact convex set (e.g., a polytope) with $0 \in \operatorname{int} A$ then $(A^\circ)^\circ = A$. For a polytope P with $0 \in \operatorname{int} A$ (Note that this implies that P is d-dimensional.), the polar P° equals $(\operatorname{Vert} P)^\circ$ and is also a d-dimensional polytope with $0 \in \operatorname{int} P^\circ$, called the *polar (or dual)* of P; and the face lattices of P and P° are anti-isomorphic.

Proof of Theorem 5.7. By Proposition 5.5, it suffices to find a point configuration $\mathcal{A}$ and a lifting function $w : \mathcal{A} \to \mathbb{R}$ such that $\mathcal{T}_w(\mathcal{A}) \cong P$. We assume that $P \subset \mathbb{R}^d$ is d-dimensional and that $0 \in \operatorname{int} P$, and we denote by $v_1, \dots, v_n$ the vertices of $P^\circ \subset \mathbb{R}^d$.

Define the point configuration $\mathcal{A} \subset \mathbb{R}^{d+1}$ as $\mathcal{A} = \{(-1, -v_1), \dots, (-1, -v_n), (-1, 0)\}$, and the lifting function $w : \mathcal{A} \to \mathbb{R}$ by $w(-1, -v_i) = 1$ for all $i = 1, \dots, n$, and $w(-1, 0) = 0$. (Since $O := (-1, 0)$ is in the interior of $\operatorname{conv}\{(-1, -v_i)\} \cong P^\circ$ the subdivision $\Sigma_w(\mathcal{A})$ is obtained by coning from O.) We get that

$$\begin{aligned}
\mathcal{E}_w(\mathcal{A}) &= \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; \begin{pmatrix} -1 & -v_1 \\ \vdots & \vdots \\ -1 & -v_n \\ -1 & 0 \end{pmatrix} x \geq -\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \right\} \\
&= \left\{ (x_1, x') \in \mathbb{R}_{\leq 0} \times \mathbb{R}^d \;\middle|\; x_1 + \langle v_i, x' \rangle \leq 1 \text{ for all } i \in [n] \right\} \\
&= \mathbb{R}_{\leq 0} \times (P^\circ)^\circ .
\end{aligned}$$

This implies that $\mathcal{T}_w(\mathcal{A}) = \{0\} \times (P^\circ)^\circ = \{0\} \times P$ and hence the theorem. □

5.3. Splits and the Split Decomposition Theorem

In Section 2.3, we defined a split to be a decomposition of a polytope with exactly two maximal cells. This definition works equally well for geometric subdivision of point configurations, but for abstract subdivisions we have to be more careful.

Example 5.8. Consider the point configuration $\mathcal{A}$ from Example 5.3; see Figure 5.1. Should the subdivisions with the maximal cells $\{1, 2, 3, 5\}$ and $\{1, 3, 4, 5\}$ and that with maximal cells $\{1, 2, 3\}$ and $\{1, 3, 4\}$ both be considered as splits?

Our Remark 2.13 and the more detailed discussion on splits of oriented matroids in Section 3.5 give the answer to this question and yield the following definition of split for a point configuration $\mathcal{A}$: If the cocircuit C of the oriented matroid $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is a split of $\mathcal{M}(\mathcal{A})$, then $S_+ = \{p \in \mathcal{A} \mid C_p \in \{0, +\}\}$ and $S_- = \{p \in \mathcal{A} \mid C_p \in \{0, -\}\}$ are the maximal cells of a split S of $\mathcal{A}$. Equivalently, splits of $\mathcal{A}$ can be defined as follows.

Lemma 5.9. *Let S be an (abstract) subdivision of $\mathcal{A}$ with exactly two maximal faces S_+ and S_-. Then the following statements are equivalent.*

(a) S is a split of $\mathcal{A}$,
(b) S is a coarsest subdivision of $\mathcal{A}$,
(c) we have $S_+ = \operatorname{conv} S_+ \cap \mathcal{A}$ and $S_- = \operatorname{conv} S_- \cap \mathcal{A}$.

After the definition of split for polytopes in Section 2.3, we remarked that for a split S there exists a hyperplane H_S that defines S and that a hyperplane H (that meets the relative interior of the polytope) defines a split if and only if it does not meet any edge of P in its relative interior. As well, for a split S of a point configuration $\mathcal{A}$ there exists as well a hyperplane H inducing a split. However, the condition has to be modified a bit: A hyperplane H defines a split of $\mathcal{A}$ if and only if it meets $\operatorname{conv} \mathcal{A}$ in its interior and for all edges E of $\operatorname{conv}\mathcal{A}$ we have that $H \cap E$ is either empty, a point of $\mathcal{A}$, or E itself. We give a complete generalization of Observation 2.12 and Proposition 3.1 to point configurations. One can give similar generalizations of the results of Section 3.1, for example for the conditions of compatibility, and so on.

Proposition 5.10. *Let $\mathcal{A}$ be a point configuration and H a hyperplane that does not intersect* $\operatorname{conv}\mathcal{A}$ *in the interior. Then the following are equivalent.*

(a) *H induces a split on $\mathcal{A}$,*
(b) *H meets all edges E of $\mathcal{A}$ in an element of $\mathcal{A}$, E, or $\emptyset$,*
(c) *H meets all faces of $\mathcal{A}$ in a face of $\mathcal{A}$ or induces a split on them,*
(d) *H meets all facets of $\mathcal{A}$ in a face of $\mathcal{A}$ or induces a split on them,*
(e) *all vertices of the subdivision of $\mathcal{A}$ with maximal faces $\mathcal{A} \cap H_+$ and $\mathcal{A} \cap H_-$ are elements of $\mathcal{A}$,*
(f) *$H \cap \operatorname{conv}\mathcal{A} = \operatorname{conv}(\mathcal{A} \cap H)$.*

Again, H_+ and H_- denote the two halfspaces in which $\mathbb{R}^{d+1}$ is divided by the hyperplane H.

An immediate consequence of this characterization is the following lemma which states that by adding point into the convex hull one cannot loose splits.

Lemma 5.11. *Let $\mathcal{A}$ be a point configuration and $\mathcal{A}' \subset \mathcal{A}$ with $\operatorname{conv}\mathcal{A} = \operatorname{conv}\mathcal{A}'$. If S is a split of $\mathcal{A}'$ with maximal faces S_+ and S_-, then $\mathcal{A}$ has a split S' with maximal faces $S'_+ = \operatorname{conv} S_+ \cap \mathcal{A}$ and $S'_- = \operatorname{conv} S_- \cap \mathcal{A}$.*

Especially, if S is a split of $\operatorname{conv}\mathcal{A}$ with maximal faces S_+ and S_-, then $\mathcal{A}$ has a split S' with maximal faces $S'_+ = S_+ \cap \mathcal{A}$ and $S'_- = S_- \cap \mathcal{A}$.

Remark 5.12. It follows from Lemma 5.11 that a two-dimensional point configuration $\mathcal{A}$ with $\operatorname{conv}\mathcal{A}$ not being a simplex cannot be unsplittable. But in contrast to the polytope case, there are a lot of different point configuration whose convex hulls are simplices. In fact, such a point configuration $\mathcal{A}$ is unsplittable if and only if there is no $p \in \mathcal{A}$ which is in the relative interior of an edge of $\operatorname{conv}\mathcal{A}$. This gives us a lot of non-trivial unsplittable two-dimensional point configurations, namely all those having a point in the relative interior but no point in the relative interior of an edge. So the simplest non-trivial unsplittable point configuration is a triangle with a point in its interior.

If one considers abstract subdivisions of $\mathcal{A}$, it does not have to be the case any more that the splits are the "simplest" possible non-trivial subdivisions. For example, in the point configuration of Example 5.3 (see Figure 5.1), the subdivision with the sole maximal cell $\{1, 2, 3, 4\}$ is non-trivial. In general, for any $p \in \operatorname{conv}(\mathcal{A} \setminus p)$ there exists a subdivision S^p of $\mathcal{A}$ with the unique maximal face $\mathcal{A} \setminus \{p\}$. (This includes configurations in convex position where one of the points occurs several times.) Such subdivisions will be called 1-*splits* (or 1-*subdivisions*); see Chapter 6 for this nomenclature. Otherwise stated, a 1-split of $\mathcal{A}$ is a subdivision with exactly one maximal face F such that $|F| = |\mathcal{A}| - 1$.

Remark 5.13. By the definition of split of a point configuration, it is clear that the set of splits of a point configuration $\mathcal{A}$ only depends on the oriented matroid of $\mathcal{A}$ as for polytopes, see Remark 2.13. By the last sentence before this remark, this is also clear for 1-splits.

LEMMA 5.14. *Splits and* 1*-splits of point configurations are regular.*

PROOF. For a split S of a point configuration $\mathcal{A}$, we have the splitting hyperplane $H_S = \operatorname{lin}(S_+ \cap S_-)$. As in the proof of Lemma 2.16, we can now define a lifting function $w_S : \mathcal{A} \to \mathbb{R}$ by Equation (2.5).

For a 1-split this is even more simple, we define a lifting function w_p by $w_p(p) = 1$ and $w_p(x) = 0$ for $x \neq p$ to get S^p. □

So we get two kinds of facets of the secondary polytope. By specializing Inequality (2.4) for the 1-split S^p of $\mathcal{A} \subset \mathbb{R}^{d+1}$, we get that this facet is simply given by

$$x_p \geq 0 . \tag{5.3}$$

For a split S, we get

$$\sum_{y \in S_+} |\langle a, p \rangle|\, x_p \geq |\langle a, c_{\operatorname{conv} S_+} \rangle|\, (d+1) \operatorname{vol}(\operatorname{conv} S_+) \tag{5.4}$$

as facet defining inequality, where a is a normal vector of H_S and $c_{\operatorname{conv} S_+}$ is the centroid of $\operatorname{conv} S_+$, as in (2.6).

As in the case of a polytope, we can define the *split polyhedron* $\operatorname{SplitPoly}(\mathcal{A})$ of a point configuration $\mathcal{A}$. It is the $(|\mathcal{A}| - d - 1)$-dimensional polyhedron in $\mathbb{R}^{|\mathcal{A}|}$ defined by the Inequalities (5.4) for the splits, the Inequalities (5.3) for the 1-splits, and the Equations (2.3). Again, the split polyhedron only depends on the oriented matroid of $\mathcal{A}$ and is bounded if $\operatorname{conv} \mathcal{A}$ is a simple polytope.

Whereas the tight span of a split is a line segment, the tight span of a 1-split only consists of the single point $O = (0, \ldots, 0)$. This implies that for the 1-split S^p the coherency index of any lifting function w with respect to w_p can be computed by the much simpler formula

$$\alpha^w_{w_p} = \min_{x \in \operatorname{Vert} \mathcal{E}_w(\mathcal{A})} \langle p, x \rangle + w(p) . \tag{5.5}$$

In fact, this means that for each maximal cell C of $\Delta_w(\mathcal{A})$ we test if $p \in C$. To formulate a split decomposition theorem for point configurations we call a lifting function $w : \mathcal{A} \to \mathbb{R}$ *split prime* if $\alpha^w_{w_S} = 0$ for all splits S and $\alpha^w_{w_p} = 0$ for all 1-splits S^p. So we get the following generalization of Theorem 2.21.

THEOREM 5.15 (Split Decomposition Theorem for Point Configurations). *Each weight function w has a coherent decomposition*

$$w = w_0 + \sum_{S^p \text{ a 1-split of } \mathcal{A}} \lambda_p w_p + \sum_{S \text{ split of } \mathcal{A}} \lambda_S w_S , \tag{5.6}$$

where w_0 is split prime, and this is unique among all coherent decompositions of w.

PROOF. The proof works in the same manner as the proof of Theorem 2.21. The splits are obtained by testing all faces F of $\Delta_w(\mathcal{A})$ where $\operatorname{aff} F$ has codimension 1, and the 1-splits are all the S^p where p is not contained in any face of $\Delta_w(\mathcal{A})$. □

If we define $\mathcal{S}(w) := \{w_s \,|\, \alpha^w_{w_S} > 0\} \cup \{w_p \,|\, \alpha^w_{w_p} > 0\}$, we can derive Corollary 2.22 literally for point configurations. Furthermore, Lemma 2.23 also holds, where $\mathcal{S}$ can be a set of splits and 1-splits for $\mathcal{A}$.

5.4. Totally Splittable Point Configurations

As in the case of polytopes, a point configuration $\mathcal{A}$ is called *totally splittable* if its secondary polytope equals its split polytope. Equivalently, $\mathcal{A}$ is totally splittable if all its (regular) subdivision are refinements of splits and 1-splits.

Before examining the general case we consider the special case of one-dimensional point configurations.

PROPOSITION 5.16. *Let $\mathcal{A}$ be one-dimensional point configuration. Then the following holds.*

(a) $\mathcal{A}$ is totally splittable.
(b) Splits of $\mathcal{A}$ are in bijection with $\mathcal{A} \cap \operatorname{relint}(\operatorname{conv} \mathcal{A})$ where multiple points are counted only once.
(c) 1-splits of $\mathcal{A}$ are in bijection with $\mathcal{A}$ where multiple points are counted with their multiplicity, but the two endpoints of $\operatorname{conv} \mathcal{A}$ are only counted if they are not single points.

PROOF. (a) It is obvious that coarsest subdivisions of one-dimensional point configurations can only have one or two maximal cells. If we have a subdivision Δ with the only maximal cell $\mathcal{A} \setminus X$, then Δ is the common refinement of the 1-splits S^p for all $p \in X$. If Δ has two maximal cells, then Δ is the refinement of the split $\Delta_{\mathcal{A}}(\Sigma(\Delta))$ and some 1-splits.

(b) Clear from the fact that the only possible splitting hyperplanes for a one-dimensional point configuration are spanned by points in the interior of $\operatorname{conv} \mathcal{A}$.

(c) A point $p \in \mathcal{A}$ can be deleted from $\mathcal{A}$ to get the 1-split $\mathcal{A}$ if and only if it is not a vertex that is not a multiple point.

□

In the case were $\mathcal{A}$ does not have any multiple points, the secondary polytope of $\mathcal{A}$ is combinatorially equivalent to a $(|\mathcal{A}| - 2)$-dimensional cube; see [**39**, Chapter 7, Section 3.A].

REMARK 5.17. Let $\mathcal{A}$ be a point configuration and $p \in \mathcal{A}$. We consider the point configuration $\mathcal{A}' := \mathcal{A} \cup \{\bar{p}\}$, where $\bar{p}$ is a copy of p, that is to say we add the point p one more time to the multiset $\mathcal{A}$. We distinguish two cases. If p is a vertex of $\operatorname{conv} \mathcal{A}$, then $\mathcal{A}'$ has exactly two more coarsest subdivisions: the 1-splits S^p and $S^{\bar{p}}$. If p is not a vertex of $\operatorname{conv} \mathcal{A}$, then $\mathcal{A}'$ has exactly one more coarsest subdivision, the 1-split $S^{\bar{p}}$. In any case, $\mathcal{A}'$ is totally splittable if and only if $\mathcal{A}$ is totally splittable. So we do not have to worry about multiple points when examining if a point configuration is totally splittable. Therefore, from now on, we only consider the case where $\mathcal{A}$ is just an (ordinary) set.

To discuss arbitrary dimensions, we give generalizations of Proposition 4.8 and Lemma 4.12 to point configurations.

PROPOSITION 5.18. *Let $\mathcal{A}$ be a totally splittable point configuration. Then each face of $\mathcal{A}$, each vertex figure, and each subconfiguration $\mathcal{A} \setminus \{v\}$ for a vertex v of* $\operatorname{conv} \mathcal{A}$ *is totally splittable. Moreover, each vertex gives rise to a vertex split.*

Similar as for polytopes, the join $\mathcal{A} * \mathcal{B}$ of a d-dimensional point configuration $\mathcal{A}$ and an e-dimensional point configuration $\mathcal{B}$ is defined as the union of $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are embedded in mutually skew affine subspaces of $\mathbb{R}^{d+e+1}$.

LEMMA 5.19. *For two point configurations $\mathcal{A}$ and $\mathcal{B}$ the join $\mathcal{A}*\mathcal{B}$ is totally splittable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are.*

The proof of these two statements works similar to the proof of Proposition 4.8 and Lemma 4.12 in Chapter 4. However, in the proof of Lemma 5.19 one has to consider splits and 1-splits instead of splits only.

Proposition 5.18 says that after removing vertices of $\operatorname{conv} \mathcal{A}$ from a totally splittable point configuration $\mathcal{A}$ the new configuration remains totally splittable. This can be generalized to the removal of arbitrary elements $p \in \mathcal{A}$.

PROPOSITION 5.20. *Let $\mathcal{A}$ be a totally splittable point configuration and $\mathcal{A}' \subset \mathcal{A}$. Then $\mathcal{A}'$ is totally splittable. Especially, if $\mathcal{A}$ is a totally splittable point configuration then* $\operatorname{conv} \mathcal{A}$ *is a totally splittable polytope.*

PROOF. Let us first consider the case that $\operatorname{conv} \mathcal{A}' \subsetneq \operatorname{conv} \mathcal{A}$. Then it follows by induction from Proposition 5.18 that $\mathcal{A} \cap \operatorname{conv} \mathcal{A}'$ is totally splittable. So we can suppose that $\operatorname{conv} \mathcal{A} = \operatorname{conv} \mathcal{A}'$.

Let Δ' be a subdivision of $\mathcal{A}'$ that is not a refinement of splits. Then $\Delta := \Delta_{\mathcal{A}}(\Sigma(\Delta'))$ is a subdivision of $\mathcal{A}$ and, by assumption, a refinement of splits and 1-splits. However, for any such split S of $\mathcal{A}$, $S' := \Delta_{\mathcal{A}'}(\Sigma(S))$ has to be a split of $\mathcal{A}'$ since $\Sigma(S)$ is a coarsening of $\Sigma(\Delta) = \Sigma(\Delta')$. So Δ' is the refinement of all such S' and possibly some 1-splits of $\mathcal{A}'$. □

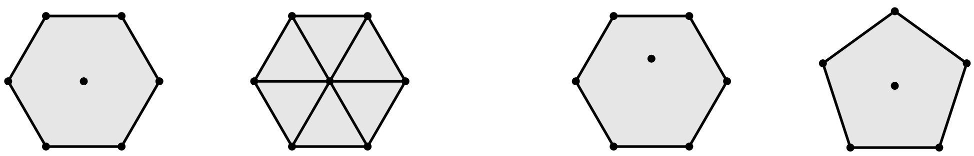

FIGURE 5.2. A totally splittable and two non-totally splittable point configurations.

EXAMPLE 5.21. (a) Let P be a regular $2n$-gon with center O. Then $\mathcal{A} := \operatorname{Vert} P \cup \{O\}$ is easily seen to be totally splittable; see Figure 5.2 (left).

(b) Let P be a regular $2n$-gon with center O and $p \neq O$ any point in the interior of P, or let P be a regular $2n+1$-gon and p any point in the interior of P. Then $\mathcal{A} := \operatorname{Vert} P \cup \{p\}$ is not totally splittable; see Figure 5.2 (right).

(c) The point configuration $\mathcal{A}$ consisting of the $2d$ vertices of the d-dimensional regular cross polytope X_d and the origin O is totally splittable. The d splits of the cross polytope (cf. Example 2.33) are split of $\mathcal{A}$ and there exists one 1-split S^O. Each d of these $d+1$ splits and 1-splits can be combined and hence we get that $\operatorname{SecPoly}(\mathcal{A}) = \operatorname{SplitPoly}(\mathcal{A})$ is a d-dimensional simplex.

In fact, the point configurations from Example 5.21 and Proposition 5.16 are the only totally splittable point configurations with points in the interior.

THEOREM 5.22. *Let $\mathcal{A}$ be a point configuration of dimension at least two. If $\mathcal{A}$ is totally splittable, then* $\operatorname{conv}\mathcal{A}$ *has the same oriented matroid as a simplex, a cross polytope, a polygon, a prism over a simplex, or a (possibly multiple) join of these polytopes, and either*

(a) $\mathcal{A} \cap \operatorname{relint}(\operatorname{conv}\mathcal{A}) = \emptyset$*,*
(b) $\mathcal{A} \cap \operatorname{relint}(\operatorname{conv}\mathcal{A}) = \{p\}$ *and* $\operatorname{conv}\mathcal{A}$ *is a $2n$-gon whose main diagonals meet in p, or*
(c) $\mathcal{A} \cap \operatorname{relint}(\operatorname{conv}\mathcal{A}) = \{p\}$ *and* $\operatorname{conv}\mathcal{A}$ *has the same oriented matroid as a cross polytope with p as center.*

Moreover, if $\mathcal{A}$ does not have any additional point in the boundary, then the converse also holds.

The *main diagonals* of a $2n$-gon are the diagonals connecting vertices with (edge graph) distance n.

PROOF. By Theorem 4.5 together with Proposition 5.20, it follows that for a totally splittable point configuration $\mathcal{A}$ the polytope $\operatorname{conv}\mathcal{A}$ has the same oriented matroid as a simplex, a cross polytope, a polygon, a prism over a simplex, or a (possibly multiple) join of these polytopes. By Example 5.21, Remark 5.17, and Proposition 5.20 together with Theorem 4.5, it follows also that if this is the case, one of the properties (a) to (c) is fulfilled and $\mathcal{A}$ does not have any additional point in the boundary, the point configuration $\mathcal{A}$ is totally splittable. It remains to show that one of the properties (a) to (c) is necessary.

Consider a totally splittable point configuration $\mathcal{A}$ and a point $p \in \mathcal{A} \cap \operatorname{relint}(\operatorname{conv}\mathcal{A})$. By coning from p, we obtain a subdivision $\Delta = \{F, F \cup \{p\} \mid F \neq \mathcal{A} \text{ a face of } \mathcal{A}\}$ of $\mathcal{A}$. By assumption, Δ has to be a refinement of splits. Now we have five different cases:

- ▹ $\operatorname{conv}\mathcal{A}$ is a simplex. It is easily seen that in this case $\mathcal{A}$ cannot have any splits (since $\mathcal{A}$ is not one-dimensional). So Δ cannot be a split subdivision.
- ▹ $\operatorname{conv}\mathcal{A}$ is an n-gon. That Δ is a split subdivision implies that for any vertex v_1 of $\operatorname{conv}\mathcal{A}$ the there has to lie a unique other vertex v_2 of $\operatorname{conv}\mathcal{A}$ on the line spanned by v_1 and p. It follows that n has to be even and that the main diagonals of $\operatorname{conv}\mathcal{A}$ has to meet in p. This is case (b); see Figure 5.2.
- ▹ $\operatorname{conv}\mathcal{A}$ is a cross polytope. That Δ is a split subdivision implies that the line segments connecting p and any vertex v of $\operatorname{conv}\mathcal{A}$ has to lie in a split hyperplane of $\operatorname{conv}\mathcal{A}$. This implies, that p is the center of $\operatorname{conv}\mathcal{A}$ and we are in case (c).
- ▹ $\operatorname{conv}\mathcal{A}$ is a prism over a $(d-1)$-dimensional simplex. We can assume that $d \geq 3$, otherwise we are in the polygon case. The maximal faces of Δ are two d-dimensional simplices and d pyramids over a prism over a $(d-2)$-dimensional simplex. However, this implies the existence of codimension-one-faces $F \in \Delta$ that comprises edges from one of the boundary simplices of $\operatorname{conv}\mathcal{A}$ to the other. So any hyperplane that contains F has to define a split of the boundary simplices, which is not possible. So there cannot be any points in the interior of $\operatorname{conv}\mathcal{A}$.

▹ $\mathrm{conv}\,\mathcal{A}$ is a join. We assume that $\mathrm{conv}\,\mathcal{A} = P_1 * P_2$. There exists codimension-two-faces of $\mathrm{conv}\,\mathcal{A}$ that are of the form $F_1 * F_2$, where F_1, F_2 are facets of P_1, P_2, respectively. So there exists a codimension-one-face F of Δ with $\mathrm{conv}\,F = \mathrm{conv}\,(F_1 * (F_2 \cup \{p\}))$. Since Δ is a split subdivision, there exists a split S of $\mathrm{conv}\,\mathcal{A}$ such that $F \subset H_S$. However, such a split cannot exists, since all splitting hyperplanes of the join $P_1 * P_2$ are of the form $\mathrm{lin}(P_1 * (H_{S_2} \cap P_2))$ or $\mathrm{lin}((H_{S_1} \cap P_1) * P_2)$, where S_1, S_2 are splits of S_1, S_2, respectively. This shows that there cannot be any points in the interior of $\mathrm{conv}\,\mathcal{A}$.

□

CHAPTER 6

Additional Facets of the Secondary Polytope

In Chapter 2, we studied the splits of a polytope P which are the subdivisions of P with two maximal cells. The natural next step is now the study of other classes of coarsest subdivisions of P. This chapter should be seen as a starting point to such an investigation.

We will begin with studying the tight spans of general coarsest subdivisions of a point configuration with a small number of maximal faces. In Section 6.2, we will consider a special class of coarsest subdivisions with k maximal cells, the k-splits, which share at least some of the properties of splits. In particular, we will prove that k-splits are regular subdivision and hence facets of the secondary polytope of P. We conclude with some open questions.

6.1. k-Subdivisions

Instead of considering only polytopes we will use the more general context of point configurations as discussed in Chapter 5. The main reason for this is, that this allows to give lower-dimensional examples. For a point configuration $\mathcal{A}$ we call an (abstract) subdivision Δ of $\mathcal{A}$ a *k-subdivision* if it has k maximal faces and cannot be coarsened non-trivially. So splits are exactly the 2-subdivisions and the notion of 1-subdivisions (which can only occur for non-convex point configurations) agrees with those of Section 5.3. By this definition, the set of all regular k-subdivisions (for all $k \geq 1$) corresponds exactly to the set of facets of the secondary polytope of $\mathcal{A}$.

For 1-subdivisions and 2-subdivisions we know exactly how the tight spans look like: they are points and line segments, respectively. However, for k-subdivisions with $k \geq 3$ the tight spans get much more complicated. We will investigate this in this section. First, we give two general statements about the tight spans of k-subdivisions.

Remark 6.1. (a) As we defined k-subdivisions so far, they are abstract subdivisions. However, we will also call the geometric subdivision $\Sigma(\Delta)$ a k-subdivision if Δ is a k-subdivision. But note that there might be another abstract subdivision Δ' that is not a k-subdivision with $\Sigma(\Delta') = \Sigma(\Delta)$. Since we are mainly interested in the tight spans of the subdivisions in this section, we will normally work with a geometric subdivisions Σ, but the results are true as well for $\Delta_{\mathcal{A}}(\Sigma)$.

(b) So far, we only defined the tight span for regular subdivisions. However, for any subdivision Σ of a point configuration $\mathcal{A}$ one can define the tight span $\mathcal{T}_\Sigma(\mathcal{A})$ as the abstract polyhedral complex that is dual to the complex of inner faces of Σ. For regular subdivisions, the usual tight span is a realization of this abstract polyhedral complex by Proposition 2.3. All we proof in this section about tight spans makes perfect sense also for non-regular subdivisions.

PROPOSITION 6.2. *Let $\mathcal{A}$ be a point configuration and Σ a k-subdivision of $\mathcal{A}$ with $k > 3$. Then the tight span $\mathcal{T}_\Sigma(\mathcal{A})$ is not a k-gon.*

PROOF. Suppose we have some subdivision Σ of $\mathcal{A}$ whose tight span is a k-gon. The k-gon corresponds to some codimension-two-face F of Σ. The facets of F are all contained in the boundary of $\operatorname{conv}\mathcal{A}$ since any facet of F that is an inner face would correspond to a three-dimensional face of $\mathcal{T}_\Sigma(\mathcal{A})$. So we have $F = \operatorname{aff} F \cap \mathcal{A}$. The edges of the k-gon are dual to codimension-one-faces of Σ whose intersection is F. We label these faces $F_1, \dots, F_k$ (We consider the indices modulo k.), where F_1 is chosen arbitrary and the others are numbered in anti-clockwise order. Furthermore, the maximal cell of Σ between F_i and F_{i+1} is called C_i. For each cell C_i we can now measure the angle α_i between the two consecutive faces F_i and F_{i+1}. Obviously, $\sum_{i=1}^{k} \alpha_i = 2\pi$, and since $k > 3$ there has to exist at least one i with $\alpha_i + \alpha_{i+1} \leq \pi$.

We now distinguish two cases. If $\alpha_i + \alpha_{i+1} = \pi$ the hyperplane $\operatorname{lin} F_i = \operatorname{lin} F_{i+2}$ defines a split of $\mathcal{A}$, contradicting the fact that Σ was supposed to be a coarsest subdivision. On the other hand, $\alpha_i + \alpha_{i+1} < \pi$ implies that $\operatorname{conv} C_i \cup \operatorname{conv} C_{i+1}$ is convex. Therefore we can construct a new subdivision Σ' of $\mathcal{A}$ with the $k-1$ maximal faces $C_1, \dots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \dots, C_k$. Since $\alpha_i + \alpha_{i+1} < \pi$, the faces $F_i, F_{i+2} \in \Sigma$ are also faces of Σ', what ensures that the intersection property holds and hence Σ' is a valid subdivision of $\mathcal{A}$. □

Note that this only shows that k-gons with $k > 3$ cannot be the sole maximal cell of a tight span of a k-subdivision. It can well be that a polygon occurs as a maximal cell of a tight span of a k-subdivision if there are other maximal cells. For the most simple example see Figure 6.2 in the top right.

For the next condition we call a polyhedral complex 2-*connected* if it is still connected if one removes any vertex (i.e., the graph of the complex is 2-connected in the usual sense).

PROPOSITION 6.3. *Let $\mathcal{A}$ be a point configuration and Σ a k-subdivision of $\mathcal{A}$. Then the tight span $\mathcal{T}_\Sigma(\mathcal{A})$ is 2-connected.*

PROOF. We will show that for a subdivision Σ of $\mathcal{A}$ whose tight span is not 2–connected there exists a subdivision Σ' of $\mathcal{A}$ that coarsens Σ.

Let v be a vertex of $\mathcal{T}_\Sigma(\mathcal{A})$ such that $\mathcal{T}_\Sigma(\mathcal{A}) \setminus \{v\}$ is not connected and T the set of vertices of some connected component of $\mathcal{T}_\Sigma(\mathcal{A}) \setminus \{v\}$. For a vertex w of $\mathcal{T}_\Sigma(\mathcal{A})$ the corresponding maximal cell of Σ is denoted by w° . We then define the new subdivision Σ' of $\mathcal{A}$ by deleting all maximal cells w° with $w \in T \cup \{v\}$ and adding $C := \bigcup_{w \in T \cup \{v\}} w^\circ$ as a maximal cell. In order that Σ' is actually a subdivision of $\mathcal{A}$, we have to show two things: First that F is convex, and second that there is no improper intersection of some face of F and some remaining face of Σ.

To prove the first assertion, assume that there exists $x, y \in \operatorname{relint} C$ such that the line segment l connecting x and y is not entirely contained in C. Then l has to intersect two codimension-one-cells C_1 and C_2 of those remaining in Σ. However, by our assumption that T is the set of vertices of some connected component of $\mathcal{T}_\Sigma(\mathcal{A}) \setminus \{v\}$ those cells has to be facets of v°. This implies that v° is not convex, a contradiction.

For the second assertion note that such an improper intersection cannot happen in the interior of $\operatorname{conv}\mathcal{A}$ since all interior faces of C are interior faces of v° by assumption. However, any improper intersection of faces F_1, F_2 in the boundary of $\operatorname{conv}\mathcal{A}$ would yield an improper intersection of some interior faces F_1', F_2' with $F_1 \subset F_1', F_2 \subset F_2'$. So Σ' is a subdivision of $\mathcal{A}$ that coarsens Σ, as desired. □

As a third condition for the tight span of a k-subdivision, we recall that any tight span of a regular subdivision has to be a contractible and hence simply connected polyhedral complex; see Section 2.2. It can be shown that this is true also for non-regular subdivisions. Additionally, this lead to the following important corollary.

Corollary 6.4. *Let $\mathcal{A}$ be a point configuration, $k \geq 3$ and Σ a k-subdivision of $\mathcal{A}$. Then all maximal faces of the polyhedral complex $\mathcal{T}_\Sigma(\mathcal{A})$ are at least two-dimensional.*

Proof. Suppose there exists some edge E in $\mathcal{T}_\Sigma(\mathcal{A})$ connecting v and w that is a maximal face. Since Σ is not a split, one of the vertices of E is strictly contained in another face of $\mathcal{T}_\Sigma(\mathcal{A})$. If we delete this vertex from $\mathcal{T}_\Sigma(\mathcal{A})$, by Proposition 6.3, the remainder is still connected. However, this implies that there has to be a path in the graph of $\mathcal{T}_\Sigma(\mathcal{A})$ connecting v with w without using E. This contradicts the simple connectedness. □

Before examining some concrete tight spans of k-subdivisions we give a strengthening of Proposition 5.5 that allows us to give examples of k-subdivisions of point configurations and have in examples for k-subdivisions of polytopes.

Proposition 6.5. *Let $\mathcal{A} \subset \mathbb{R}^{d+1}$ be a point configuration and $\Sigma_w(\mathcal{A})$ a regular k–subdivision of $\mathcal{A}$. Then there exists a polytope $P \subset \mathbb{R}^{d+2}$ and a regular k-subdivision $\Sigma_{w'}(P)$ of P such that $\mathcal{T}_{w'}(P)$ is affinely isomorphic to $\mathcal{T}_w(\mathcal{A})$.*

Proof. We use the same construction as in the proof of Proposition 5.5. What remains to show is that the subdivision $\Sigma_{w'}(P)$ cannot be refined non-trivially. Suppose there exists some non-trivial coarsening Σ of $\Sigma_{w'}(P)$. It is easily checked that $\Sigma' := \{C \cap (\mathbb{R}^{d+1} \times \{0\}) \,|\, C \in \Sigma\}$ is a subdivision of $\mathcal{A} \times \{0\}$, so we have a subdivision of $\mathcal{A}$ that coarsens $\Sigma_w(\mathcal{A})$ non-trivially. This contradicts the assumption that $\Sigma_w(\mathcal{A})$ was a k-subdivision. □

No we will examine the tight spans of k-subdivision for small k.

Lemma 6.6. *Let $\mathcal{A}$ be a point configuration, and Σ a 3-subdivision of $\mathcal{A}$. Then the tight span of Σ is a triangle.*

Proof. Obviously, the only simple connected polyhedral complexes with three points are a triangle or two line segments connected at one point. However, the latter cannot occur by Proposition 6.4. □

Example 6.7. Let P the bipyramid over a triangle. It is easily seen, that P has two coarsest subdivision which are both also triangulations: First the split subdivision obtained by the only split which yield two congruent tetrahedra, second the subdivision obtained by taking the diagonal between the two non-adjacent vertices and forming three tetrahedra around it. The latter is a 3-subdivisions whose tight span is a triangle.

Lemma 6.8. *Let $\mathcal{A}$ be a point configuration, and Σ a 4-subdivision of $\mathcal{A}$. Then the tight span of Σ is either a tetrahedron, consists of three triangles with a common vertex, or consists of two triangles glued together at one edge.*

Proof. We are searching for simply connected polyhedral complexes with four vertices. By Corollary 6.4, we have the additional condition that all maximal cells has to be at least two-dimensional. So the candidates are a tetrahedron, two triangles glued together at one edge, three triangles with a common vertex or a quadrangle. However, the quadrangle cannot occur by Proposition 6.2. □

Example 6.9. In Figure 6.1, we depict examples of 4-subdivisions of point configurations together with their tight spans which are the two two-dimensional complexes from Lemma 6.8. To get a tetrahedron as tight span, take as point configuration a tetrahedron with an inner point and cone from this point. This subdivision is a 4-split, as discussed in the next section. Note that by Proposition 6.5 there also exist polytopes with these tight spans.

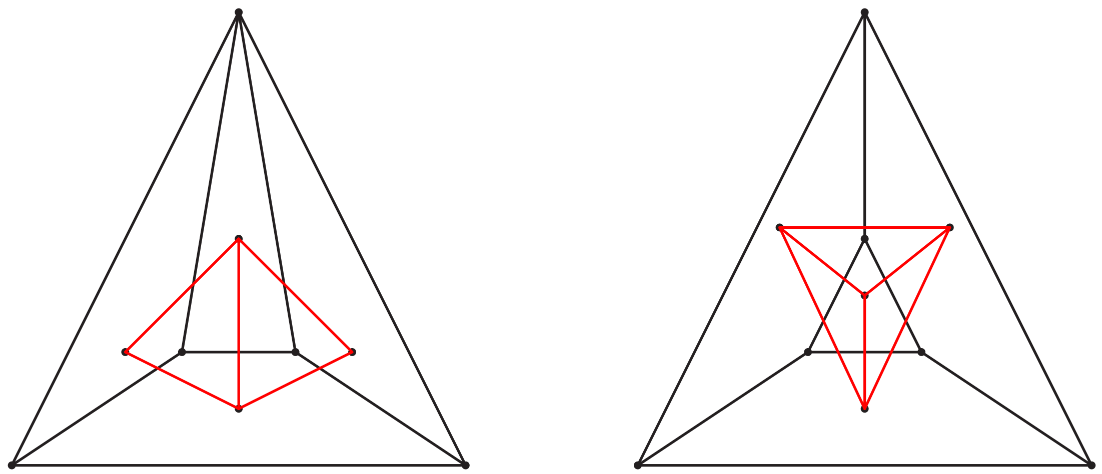

Figure 6.1. 4-subdivisions and their tight spans.

Lemma 6.10. *Let $\mathcal{A}$ be a point configuration and Σ a 5-subdivision of $\mathcal{A}$. Then the tight span of Σ cannot consist of a quadrangle and a triangle glued together at one edge.*

Proof. Suppose there exists a point configuration $\mathcal{A}$ and a subdivision Σ with such a tight span. We can now argue as in the proof of Proposition 6.2 by letting F be the face of Σ dual to the quadrangle. We adopt the notion from the proof of Proposition 6.2. The only case that is not covered by the argumentation there, is that where $\alpha_i + \alpha_{i+1} < \pi$, and C_i and C_{i+1} are the cells corresponding to the vertices of the edge of the tight span which is the intersection of the quadrangle and the triangle. However, in this case, we simply do not take $C_i \cup C_{i+1}$ as a new maximal cell, but $C := C_i \cup C_{i+1} \cup C^{\star}$, where $C^{\star}$ is the cell of Δ corresponding to the unique non-quadrangle vertex of the tight span. One now directly sees that C is convex and the intersection property for the subdivision holds by the same argumentation as in the proof of Proposition 6.2. □

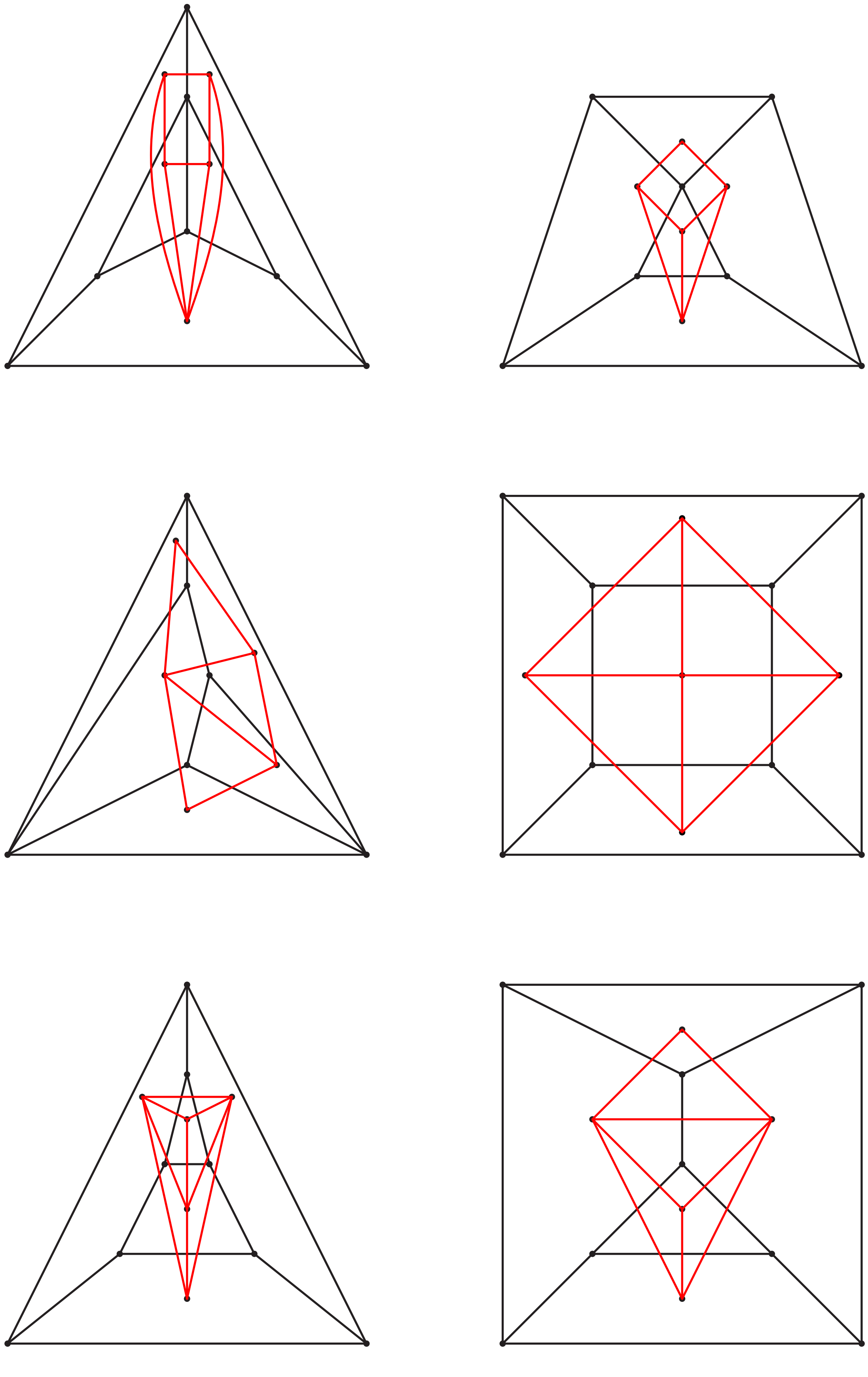

FIGURE 6.2. The 5-subdivisions with planar tight spans.

Example 6.11. In Figure 6.2, we depict examples of 5-subdivisions covering all planar tight spans that may occur. For the two topmost subdivisions it has to be carefully checked that these are really coarsest subdivisions what is true because all suitable unions are not convex.

Example 6.12. In Figure 6.3, we depict examples of 5-subdivisions with pure three-dimensional tight spans. The first tight span is a pyramid, and the subdivision is obtained by taking as point configuration the vertices of another pyramid P together with any inner point v and take as maximal simplices the cones from v over all facets of P. (This is the same construction as in the proof of Theorem 5.7; pyramids are self-dual.) To the left, we have as tight span a bipyramid over a triangle, which is obtained in the same way by taking a prism over a triangle with one inner point. The tight span of the subdivision to the right of Figure 6.3 consist of two tetrahedra glued at a facet. To get it, take a prism over a simplex with two inner points connected by an edge. In the same way, one could take three inner point in a plane parallel to the top and bottom facets, to get a 5-subdivision whose tight span consists of three tetrahedra all sharing an edge. Taking as point configuration the vertices of two simplices, one of them in the interior of the other, one can get a 5-subdivision whose tight span consists of four tetrahedra all sharing a vertex. Altogether, we have described all pure three-dimensional complex that may occur as the tight span of a 5-subdivision.

Example 6.13. An example of a subdivision with non-pure tight span is given in Figure 6.4 (left). Its tight span is a tetrahedron with a triangle glued at an edge. The point configuration $\mathcal{A}$ consists of the six vertices of an octahedron together with an interior point. (Note that the inner point cannot be chosen arbitrarily in this case to get a coarsest subdivision.) The subdivision Σ of $\mathcal{A}$ with maximal faces $\{2,3,4,5,7\}$, $\{1,2,5,7\}$, $\{1,3,5,7\}$, $\{2,3,4,6\}$, and $\{1,2,3,6,7\}$ can be show to be coarsest and its tight span is as desired, as can be seen from Figure 6.4. Our last example is a 5-subdivision with a two-dimensional tight span that is not planar. In Figure 6.4 (right), we depicted a polytope subdivided into three simplices and one (rotated) prism over a triangle. Reflecting this complex at the hexagonal facet, one arrives at a polytope with twelve vertices subdivided into six simplices and two triangular prism. The union of each pair of simplices is convex, hence we can replace them by their union, arriving at a 5-subdivision. The tight span of this 5-subdivision consists of three triangles that share a common edge.

Remark 6.14.

(a) In the proof of Lemma 6.10, we have the first case of a simply connected polyhedral complex that cannot occur as a tight span of a k-subdivision and is not excluded by Proposition 6.2 or Proposition 6.3.
(b) The examples in Figure 6.2 show that all simply connected and 2-connected polyhedral complex with five vertices whose maximal faces are all triangles can occur as the tight span of some point configuration. In fact, it can be shown that this true for such complexes with an arbitrary number of vertices.
(c) The proof of Lemma 6.10 can be extended to show that the tight span of any k-subdivision cannot be a $(k-1)$-gon glued with a triangle.

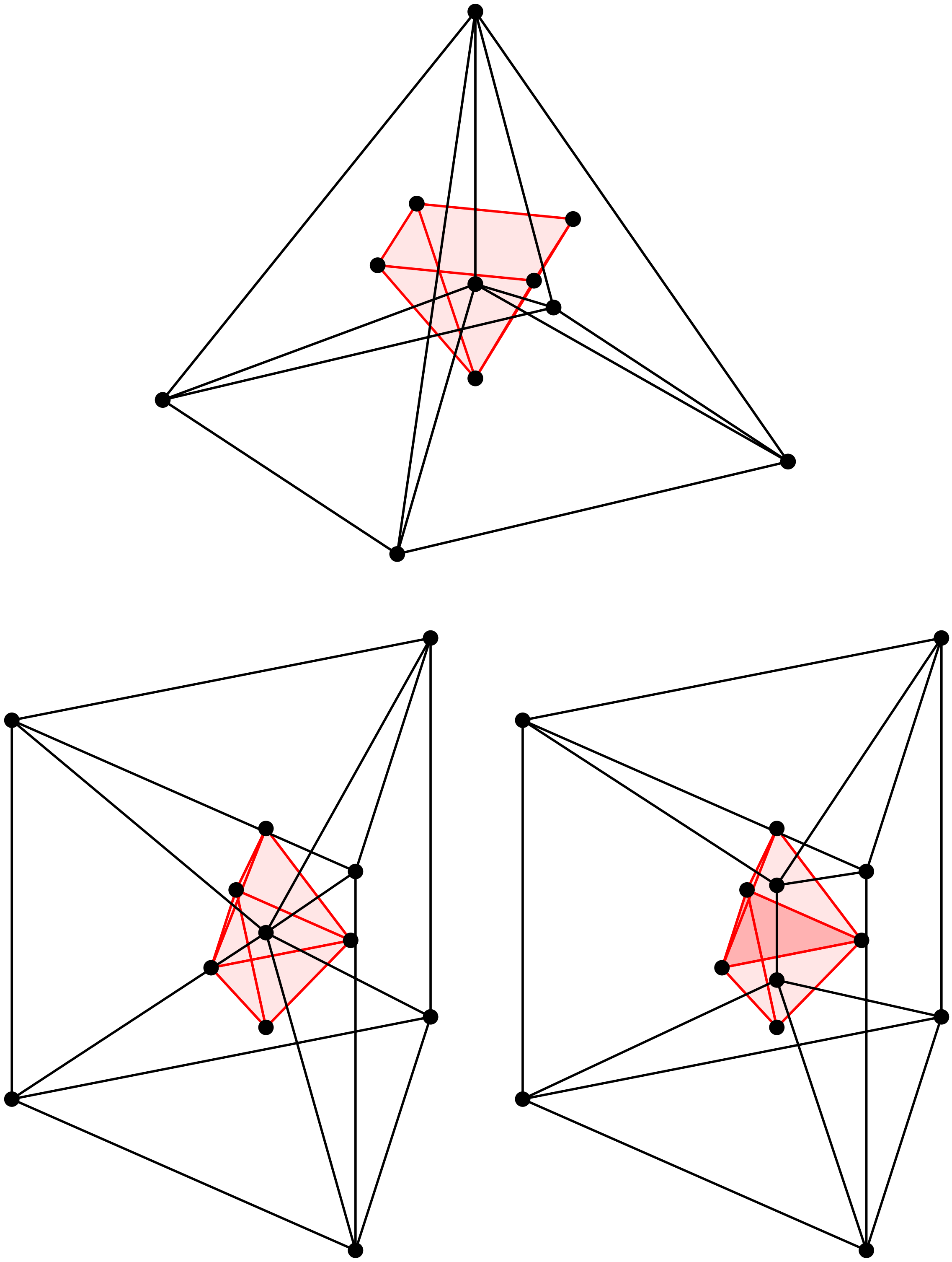

FIGURE 6.3. Some 5-subdivisions with pure three-dimensional tight spans.

As we have seen in Lemma 6.8 and Lemma 6.10, all three-dimensional polytopes with up to five vertices can appear as tight spans of k-subdivisions. Since all polytopes can occur as the tight span of some subdivision by Theorem 5.7 it seem natural to ask if all polytopes of dimension three or higher can occur as the tight span of some k-subdivision. The following proposition answers this question negatively.

PROPOSITION 6.15. *Not all polytopes with dimension three or higher can occur as tight spans of k-subdivisions. Especially, a prism over a triangle cannot occur as the tight span of a* 6*-subdivision.*

PROOF. Suppose there exists some point configuration $\mathcal{A}$ and a subdivision Σ of $\mathcal{A}$ such that $\mathcal{T}_\Sigma(\mathcal{A})$ is a prism over a triangle. Denote by F the codimension-three-cell of Σ corresponding to the prism itself, and by F_1, F_2, F_3 the codimension-one-cells corresponding to the three parallel edges of $\mathcal{T}_\Sigma(\mathcal{A})$. Since $F = F_1 \cap F_2 \cap F_3$ is of codimension two in the F_i, either F_1, F_2, and F_3 lie in a common hyperplane H, or for each of the hyperplanes H_i spanned by one of the F_i the relative interiors of the other two lie on the same side of H_i. In the first case, the hyperplane H defines a split of $\mathcal{A}$, since the intersection of H with the boundary of $\operatorname{conv}\mathcal{A}$ equals the intersection of $F_1 \cup F_2 \cup F_3$ with the boundary, and hence cannot produce additional vertices. Obviously, this split coarsens Σ.

In the second case, we denote by H_i^+ the one of the two (closed) halfspaces defined by H_i that contains the two other faces. Obviously, $C := \operatorname{conv}\mathcal{A} \cap H_1^+ \cap H_2^+ \cap H_3^+$ is convex and the union of three maximal cells of Σ. So we can define a new subdivision Σ' of $\mathcal{A}$ by replacing these three cells with C. Sine F_1, F_2, and F_3 are facets of C, there also cannot any non-trivial intersections, either, so Σ' is a valid subdivision that coarsens Σ.

Altogether, Σ cannot be a coarsest subdivision of $\mathcal{A}$ and hence no 6-subdivision. □

6.2. k-Splits

As we have seen in the previous section, if k grows large there exist a lot of different possible tight spans for k-subdivisions. So an investigation of general k-subdivisions might be quite complicated. However, in this section we will investigate for each k a special class of k-subdivisions which behave similar to splits. We call a k-subdivision Δ of $\mathcal{A}$ a *k-split* if Δ has an inner face of codimension $k-1$.

It is easily seen that Δ is a k-split if and only if the tight span $\mathcal{T}_\Delta(\mathcal{A})$ is a $(k-1)$-dimensional simplex. All k-subdivisions are k-splits for $k \in \{1, 2, 3\}$. Especially, the splits are the 2-splits.

EXAMPLE 6.16. An example of a k-split is given by taking a $(k-1)$-dimensional simplex with a point in the interior and coning from that point. For an example of a polytope (with less vertices than that one could obtain from Proposition 6.5), one can take a bipyramid over a $(k-1)$-dimensional simplex and cone from the edge connecting the two pyramid vertices.

As to each split there corresponds a unique hyperplane, to each k-split (for $k > 1$) there corresponds a unique subspace of codimension $k-1$. However, for splits we also have the property that if a hyperplane H defines a split, this split is uniquely

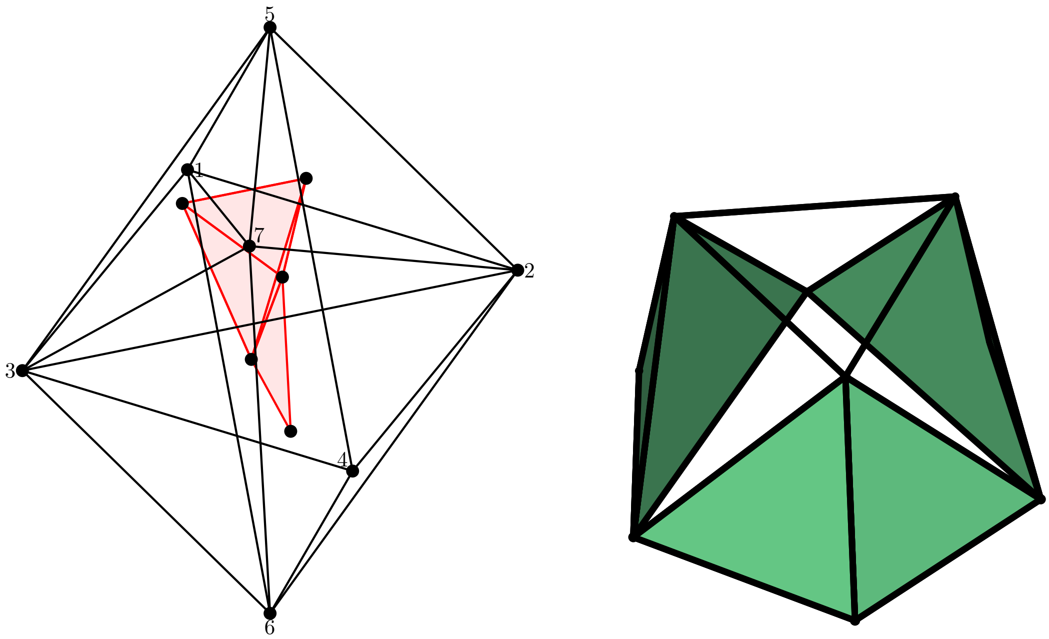

FIGURE 6.4. Two 5-subdivisions and their tight spans.

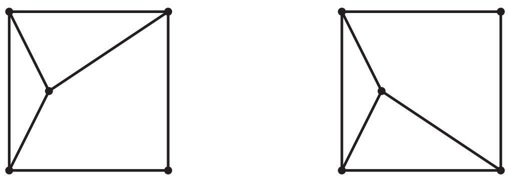

FIGURE 6.5. A point configuration with a codimension-two-face (the interior point) that corresponds to two different 3-splits.

determined by H. This does not hold any more for k-splits with $k \geq 3$; see Figure 6.5 and Example 6.19.

In Proposition 5.10, we saw that a hyperplane H defines a split of a point configuration if and only if it meets all edges E of $\mathcal{A}$ in a an element of $\mathcal{A}$, E, or the empty set. One direction of this generalizes to k-splits as follows.

PROPOSITION 6.17. *If U is the unique codimension-$(k-1)$-subspace corresponding to some k-split of a point configuration $\mathcal{A}$, then the following equivalent conditions are satisfied.*

(a) U meets all faces F of $\mathcal{A}$ with $\dim F \leq k-1$ in a face of $\mathcal{A}$ or corresponds to an l-split on them with $l \leq k$,
(b) U meets all faces of $\mathcal{A}$ in a face of $\mathcal{A}$ or corresponds to an l-split on them for some $l \leq k$,
(c) U meets all facets of $\mathcal{A}$ in a face of $\mathcal{A}$ or corresponds to an l-split on them for some $l \leq k$.

Proof. First one sees, that if Δ is a k-split of $\mathcal{A}$, the induced subdivision to each face of $\mathcal{A}$ has to be an l-split for $l \leq k$, or the trivial subdivision. This implies that all conditions has to be satisfied. That (a) implies (b) follows from the fact that if a codimension-$(k-1)$-subspace U intersects some face F with $\dim F \geq k$ in its interior, the subspace U has to intersect some of the faces of F of dimension $k-1$. The (c) is also equivalent follows by applying the equivalence of (a) and (b) to $\mathcal{A}$ and its facets. □

However, in contrast to the split case, the converse of Proposition 6.17 does not hold if $k \geq 3$. For an example, consider the polytope depicted in Figure 6.6. The codimension-two-subspace spanned by the top and bottom vertices does not correspond to any 3-split.

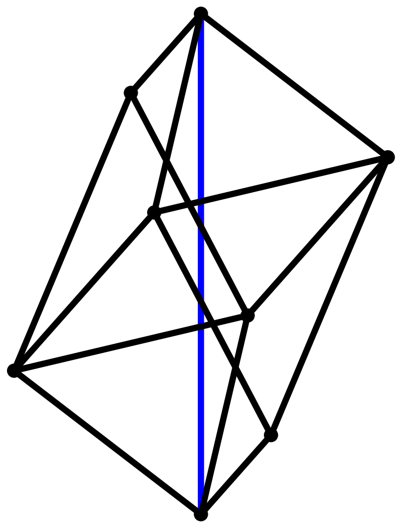

Figure 6.6. A polytope with an inner edge that does not correspond to a 3-split.

The most important property of splits is shared by the k-splits: They all are regular subdivisions.

Theorem 6.18. *All k-splits are regular.*

Proof. Let $\mathcal{A}$ be a d-dimensional point configuration and Δ a k-split of P. So Δ has a unique interior face F of dimension $d-(k-1)$. We can assume that the origin is contained in $\operatorname{conv} F$. We now project Δ onto the subspace orthogonal to F and obtain a subdivision Δ' of the $(k-1)$-dimensional point configuration $\operatorname{im}\mathcal{A}$ with the origin as an interior vertex. If we now take for each face F of Δ' the cone spanned by F we get a polyhedral fan subdividing $\mathbb{R}^{k-1}$. The dual complex of this fan is isomorphic to the tight span of Δ' and Δ. For each of the k rays r_i of this fan (which correspond to interior faces of dimension $d-k+2$ of Δ) we take a vector e_i of length one that spans this ray. Now for each vertex $p \in \mathcal{A}$ we define the weight of p as $\sum_{i=1}^{k-1} \lambda_i$ where the image p' of p under the projection is generated by the positive combination $p' = \sum_{i=1}^{k-1} \lambda_i e_i$ where only those e_i are positive for which the cone that contains a' is generated by r_i. This lifting function defines Δ. □

By setting the weight w as in the proof of Theorem 6.18, one gets an explicit description of the corresponding facet of the secondary polytope by specializing Equation (2.4). Furthermore, one could define the *k-split polyhedron* $k - \mathrm{SplitPoly}(\mathcal{A})$ of a point configuration $\mathcal{A}$ by taking those inequalities for all l-splits with $l \leq k$ together with the Equations (2.3), and the *semi-split polyhedron* $\mathrm{semi} - \mathrm{SplitPoly}(\mathcal{A})$ by taking the inequalities for all k. Obviously, $\mathrm{SplitPoly}(\mathcal{A}) \supset k{-}\mathrm{SplitPoly}(\mathcal{A}) \supset \mathrm{semi} - \mathrm{SplitPoly}(\mathcal{A}) \supset \mathrm{SecPoly}(\mathcal{A})$, so we get new outer "approximations" for the secondary polytope.

We call a polytope P (or a point configuration $\mathcal{A}$) *totally k-splittable* if all regular subdivisions of P are common refinements of l-splits with $l \leq k$. We call P *totally semi-splittable* if P is totally k-splittable for some k, or, equivalently, all regular subdivisions of P are common refinements of k-splits for some k. It is easily seen, that a d-dimensional polytope (or a $(d-1)$-dimensional point configuration) is totally semi-splittable if and only if it is totally d-splittable.

Example 6.19. The 3-cube C_3 is totally 3-splittable, hence totally semi-splittable. It has 14 splits (see Example 2.19), and eight 3-splits: Each diagonal of the cube corresponds to two 3-splits by subdividing C_3 into three square pyramids with one of the vertices of the diagonal as apex. By adding the eight inequalities corresponding to this 3-splits to the split polytope of C_3, we obtain a new computation of the secondary polytope of C_3, verifying the results of [**75**].

Example 6.20. The 4-cube C_4 is not totally semi-splittable. The secondary polytope of C_4 has $80,876$ facets that come in 334 orbits (see [**53**]). Four of these orbits are splits (see Example 2.20), five are 3-splits, and three are 4-splits.

One might ask whether there exists some generalization of the Split Decomposition Theorem 2.21 to point configurations. For k-splits with arbitrary k, this is obviously false, since, for example, the subdivision of the 3-cube to the left of Figure 2.3 can be obtained as the common refinement of either three splits or two 3-splits (see also Example 2.34). However, even if one fixes $k \geq 3$, no similar result can be valid: The triangulation to the left of Figure 6.7 can be obtained as the common refinement of the 3-split A and either of the two 3-splits B_1, B_2 in Figure 6.7.

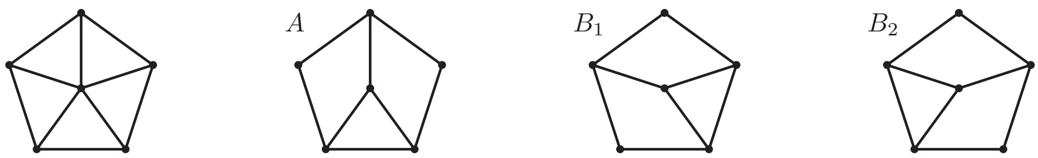

Figure 6.7. There is no unique 3-split decomposition.

A set $\mathcal{S}$ of k-splits (for either fixed or arbitrary k) is called *weakly compatible* if there exists a common refinement of all $\Sigma \in \mathcal{S}$. It is called *compatible*, if none of the codimension-one-faces of two different Σ_1, Σ_2 intersect in the interior of $\mathrm{conv}\,\mathcal{A}$. A subdivision is called a *k-split subdivision*, if it is induced by a weakly compatible system of l-splits with $l \leq k$. In this way, also a generalization of the split complex can be defined. It will still be a flag simplicial complex as in Proposition 2.25. However, one has to be careful in defining an analogue of $\mathrm{Split}^{\mathrm{w}}(\mathcal{A})$ since this will not be a simplicial complex any more.

6.3. Open Questions and Concluding Remarks

We have discussed some conditions on when polyhedral complexes can be the tight span of some k-subdivision. However, we also gave examples that these conditions are not sufficient. For complexes with a sole maximal cell, we showed that the only possibility in dimension two is a triangle, and that in dimension three not all polytopes may occur. This naturally leads to the following question.

Question 6.21. Which polyhedral complexes, especially, which polytopes occur as tight spans of k-subdivisions?

It seams that k-splits are a very natural generalization of splits. In Chapter 4, we classified all totally splittable polytopes.

Question 6.22. Which polytopes are totally k-splittable? Which polytopes are totally semi-splittable?

The answer to this question might lead to interesting new classes of polytopes, the class of all totally 3-splittable polytopes, all totally 4-splittable polytopes, and so on. This would help to get new insights in the structure of secondary polytopes. Especially, since for the class of totally 2-splittable polytopes all secondary polytopes are known (see Example 4.6), a classification of totally k-splittable polytopes for small $k \geq 3$ could lead to explicit computations of some secondary polytopes.

CHAPTER 7

How To Draw Tropical Planes

This chapter is joint work with Anders Jensen, Michael Joswig, and Bernd Sturmfels [**45**].

7.1. Introduction

A line in tropical projective space $\mathbb{TP}^{n-1}$ is an embedded metric tree which is balanced and has n unbounded edges pointing into the coordinate directions. The parameter space of these objects is the tropical Grassmannian $\mathrm{Gr}(2,n)$. This is a simplicial fan [**84**], known to evolutionary biologists as the *space of phylogenetic trees* with n labeled leaves [**73**, Section 3.5], and known to algebraic geometers as the *moduli space of rational tropical curves* [**70**].

Speyer [**83, 82**] introduced higher-dimensional tropical linear spaces. They are contractible polyhedral complexes all of whose maximal cells have the same dimension $d-1$. Among these are the realizable tropical linear spaces which arise from $(d-1)$-planes in classical projective space $\mathbb{P}^{n-1}_{\mathbb{K}}$ over a field $\mathbb{K}$ with a non-archimedean valuation. Realizable linear spaces are parameterized by the tropical Grassmannian $\mathrm{Gr}(d,n)$, as shown in [**84**]. Note that all trees ($d = 2$) are realizable. Tropical Grassmannians represent compact moduli spaces of hyperplane arrangements. Introduced by Alexeev, Hacking, Keel, and Tevelev [**1, 44, 61**], these objects are natural generalizations of the moduli space $\overline{M}_{0,n}$.

In this chapter, we focus on the case $d = 3$. By a *tropical plane* we mean a two-dimensional tropical linear subspace of $\mathbb{TP}^{n-1}$. It was shown in [**84**, Section 5] that all tropical planes are realizable when $n \leq 6$. This result rests on the classification of planes in $\mathbb{TP}^5$ which is shown in Figure 7.1. We here derive the analogous complete picture of what is possible for $n = 7$. In Theorem 7.10, we show that for larger n most tropical planes are not realizable.

Tropical linear spaces are represented by vectors of Plücker coordinates. The axioms characterizing such vectors were discovered two decades ago by Andreas Dress who called them *valuated matroids*. We therefore propose the name *Dressian* for the tropical prevariety $\mathrm{Dr}(d,n)$ which parameterizes $(d-1)$-dimensional tropical linear spaces in $\mathbb{TP}^{n-1}$.

This paper is organized as follows. In Section 2, we review the formal definition of the Dressian and the Grassmannian, and we present our results on $\mathrm{Gr}(3,7)$ and $\mathrm{Dr}(3,7)$. These also demonstrate the remarkable scope of current software for tropical geometry.

Tropical planes are dual to regular matroid subdivisions of the hypersimplex $\Delta(3,n)$. The theory of these subdivisions is developed in Section 3, after a review of matroid basics, and this allows us to prove various combinatorial results about the Dressian $\mathrm{Dr}(3,n)$.

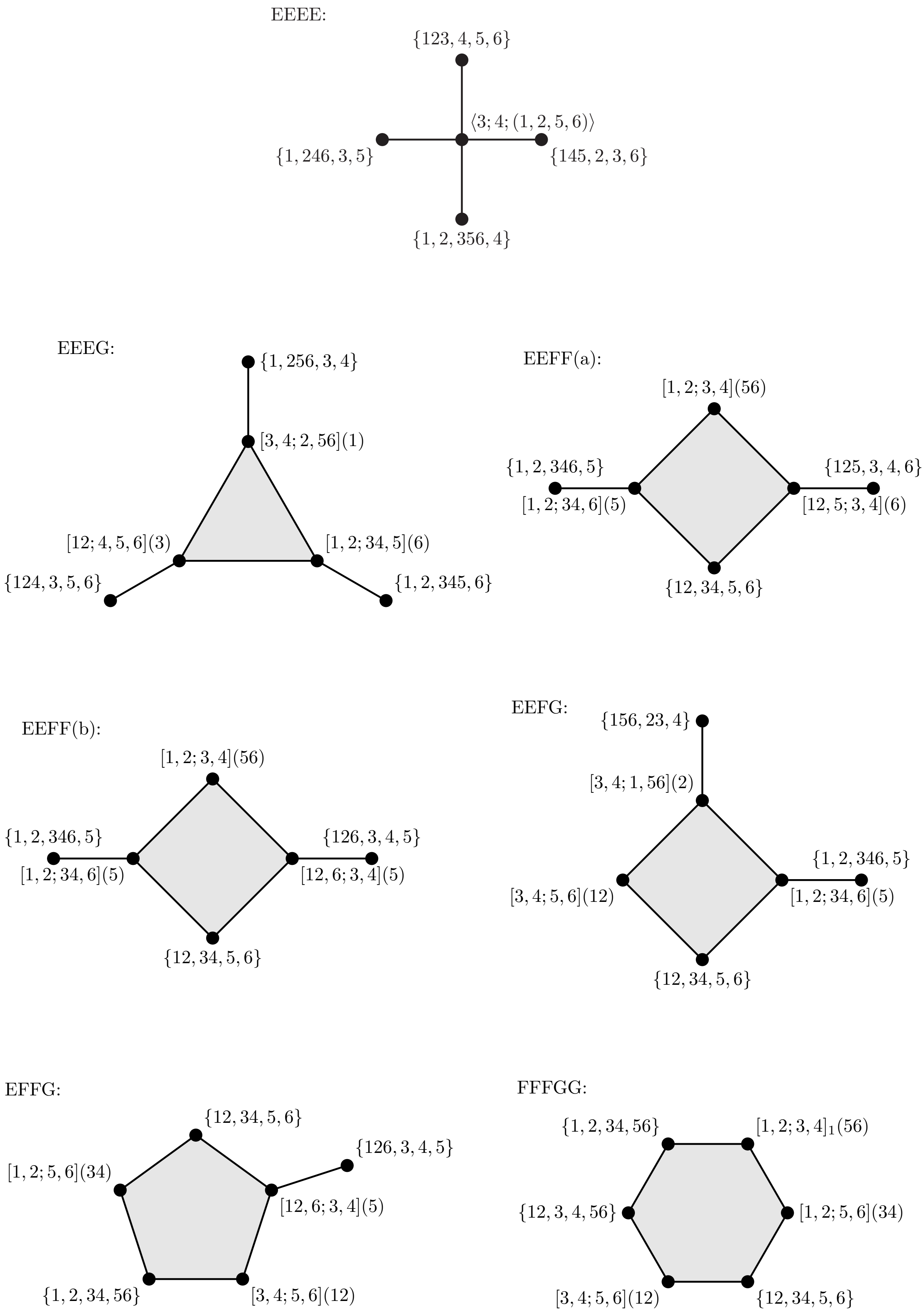

FIGURE 7.1. The seven types of generic tropical planes in $\mathbb{TP}^5$.

A main contribution is the bijection between tropical planes and arrangements of metric trees in Theorem 7.15. This bijection tropicalizes the following classical picture. Every plane in $\mathbb{P}^{n-1}_{\mathbb{K}}$ corresponds to an arrangement of n lines in $\mathbb{P}^2_{\mathbb{K}}$, and hence to a rank-three-matroid on n elements. Lines are now replaced by trees, and arrangements of trees are used to encode matroid subdivisions. These can be non-regular, as shown in Section 7.4. Section 7.5 answers the question in the title of this paper, and, in particular, it explains the seven diagrams in Figure 7.1 and their 94 analogs for $n = 7$. In Section 7.6, we extend the notion of Grassmannians and Dressians from $\Delta(k, n)$ to arbitrary matroid polytopes.

We are indebted to Francisco Santos, David Speyer, Walter Wenzel, and Lauren Williams for various comments.

7.2. Computations

Let I be a homogeneous ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_t]$ over a field $\mathbb{K}$. Each vector $\lambda \in \mathbb{R}^t$ gives rise to a partial term order and thus defines an initial ideal $\mathrm{in}_\lambda(I)$, by choosing terms of lowest weight for each polynomial in I. The set of all initial ideals of I induces a fan structure on $\mathbb{R}^t$. This is the *Gröbner fan* of I, which can be computed using `Gfan` [**56**]. The subfan induced by those initial ideals which do not contain any monomial is the *tropical variety* $\mathbb{T}(I)$. If I is a principal ideal then $\mathbb{T}(I)$ is a *tropical hypersurface.* A *tropical prevariety* is the intersection of finitely many tropical hypersurfaces. Each tropical variety is a tropical prevariety, but the converse does not hold [**78**, Lemma 3.7].

Consider a fixed $d \times n$-matrix of indeterminates. Then each $d \times d$-minor is defined by selecting d columns $\{i_1, i_2, \ldots, i_d\}$. Denoting the corresponding minor $p_{i_1 \ldots i_d}$, the algebraic relations among all $d \times d$-minors define the *Plücker ideal* $I_{d,n}$ in $\mathbb{K}[p_S]$, where S ranges over $\binom{[n]}{d}$, the set of all d-element subsets of $[n] := \{1, 2, \ldots, n\}$. The ideal $I_{d,n}$ is a homogeneous prime ideal. The *tropical Grassmannian* $\mathrm{Gr}(d, n)$ is the tropical variety of the Plücker ideal $I_{d,n}$. Among the generators of $I_{d,n}$ are the *three term Plücker relations*

$$p_{Sij} p_{Skl} - p_{Sik} p_{Sjl} + p_{Sil} p_{Sjk}\,, \tag{7.1}$$

where $S \in \binom{[n]}{d-2}$ and $i, j, k, l \in [n] \backslash S$ are pairwise distinct. Here Sij is shorthand notation for the set $S \cup \{i, j\}$. The relations (7.1) do not generate the Plücker ideal $I_{d,n} \subset \mathbb{K}[p_S]$ for $d \geq 3$, but they always suffice to generate the image of $I_{d,n}$ in the Laurent polynomial ring $\mathbb{K}[p_S^{\pm 1}]$.

The *Dressian* $\mathrm{Dr}(d, n)$ is the tropical prevariety defined by all three term Plücker relations. The elements of $\mathrm{Dr}(d, n)$ are the *finite tropical Plücker vectors* of Speyer [**83**]. A *general tropical Plücker vector* is allowed to have ∞ as a coordinate, while a *finite* one is not. The three term relations define a natural *Plücker fan structure* on the Dressian $\mathrm{Dr}(d, n)$: two weight vectors λ and λ' are in the same cone if they specify the same initial form for each trinomial (7.1). In Sections 3 and 4, we shall derive an alternative description of the Dressian $\mathrm{Dr}(d, n)$ and its Plücker fan structure in terms of matroid subdivisions.

It is clear from the definitions that the Dressian contains the tropical Grassmannian (over any field $\mathbb{K}$) as a subset of $\mathbb{R}^{\binom{n}{d}}$; but it is far from obvious how the fan structures are related. Results of [**84**] imply that the fans $\mathrm{Gr}(2,n)$ and $\mathrm{Dr}(2,n)$ are equal and that $\mathrm{Gr}(3,6) = \mathrm{Dr}(3,6)$ as sets. Using computations with the software systems `Gfan` [**56**], `homology` [**31**], `Macaulay2` [**41**], and `polymake` [**37**] we obtained the following results about the next case $(d,n) = (3,7)$.

THEOREM 7.1. *Fix any field* $\mathbb{K}$ *of characteristic different from two. The tropical Grassmannian* $\mathrm{Gr}(3,7)$*, with its induced Gröbner fan structure, is a simplicial fan with* f*-vector*

$$(721, 16800, 124180, 386155, 522585, 252000).$$

The homology of the underlying five-dimensional simplicial complex is free Abelian, and it is concentrated in top dimension:

$$H_*(\mathrm{Gr}(3,7);\mathbb{Z}) \;=\; H_5(\mathrm{Gr}(3,7);\mathbb{Z}) \;=\; \mathbb{Z}^{7470}.$$

The result on the homology is consistent with Hacking's theorem in [**43**, Theorem 2.5]. Inspired by Markwig and Yu [**69**], we conjecture that the simplicial complex $\mathrm{Gr}(3,7)$ is shellable.

THEOREM 7.2. *The Dressian* $\mathrm{Dr}(3,7)$*, with its Plücker fan structure, is a non-simplicial fan. The underlying polyhedral complex is six-dimensional with* f*-vector*

$$(616, 13860, 101185, 315070, 431025, 211365, 30).$$

Its 5*-skeleton is triangulated by the Grassmannian* $\mathrm{Gr}(3,7)$*, and the homology is*

$$H_*(\mathrm{Dr}(3,7);\mathbb{Z}) \;=\; H_5(\mathrm{Dr}(3,7);\mathbb{Z}) \;=\; \mathbb{Z}^{7440}.$$

The Grassmannian and the Dressian were defined as fans in $\mathbb{R}^{\binom{n}{d}}$. One could also view them as subcomplexes in the *tropical projective space* $\mathbb{TP}^{\binom{n}{d}-1}$, which is the compact space obtained by taking $(\mathbb{R}\cup\{\infty\})^{\binom{n}{d}}\backslash\{(\infty,\ldots,\infty)\}$ modulo tropical scalar multiplication. We adopt that interpretation in Section 7.6. Until then, we stick to $\mathbb{R}^{\binom{n}{d}}$ but we take its quotient modulo the common n-dimensional lineality space of both fans. This gives pointed fans in $\mathbb{R}^{\binom{n}{d}-n}$. We represent these as polyhedral complexes in the sphere of dimension $\binom{n}{d} - n - 1$. The resulting polyhedral complex $\mathrm{Gr}(d,n)$ is $(d(n-d)-n)$-dimensional, and $\mathrm{Dr}(d,n)$ is a generally higher-dimensional polyhedral complex whose support contains the support of $\mathrm{Gr}(d,n)$. These are the polyhedral complexes referred to in Theorems 7.1 and 7.2. For instance, $\mathrm{Gr}(2,5) = \mathrm{Dr}(2,5)$ is the Petersen graph.

We note that the combinatorial and algebraic notions in this paper are compatible with the geometric theory developed in Mikhalkin's book [**70**]. We here use "min" for tropical addition, the set $\mathbb{T}^{k-1} = \mathbb{R}^k/\mathbb{R}(1,1,\ldots,1)$ is the *tropical torus*, and the tropical projective space $\mathbb{TP}^{k-1}$ is a compactification of $\mathbb{T}^{k-1}$ which is a closed simplex.

The symmetric group Sym_7 acts naturally on both $\mathrm{Gr}(3,7)$ and $\mathrm{Dr}(3,7)$, and it makes sense to count their cells up to this symmetry. The face numbers modulo Sym_7 are

$$f(\mathrm{Gr}(3,7) \bmod \mathrm{Sym}_7) \;=\; (6,37,140,296,300,125) \quad \text{and}$$
$$f(\mathrm{Dr}(3,7) \bmod \mathrm{Sym}_7) \;=\; (5,30,107,217,218,94,1).$$

Thus the Grassmannian $\mathrm{Gr}(3,7)$ modulo Sym_7 has 125 five-dimensional simplices, and these are merged to 94 five-dimensional polytopes in the Dressian $\mathrm{Dr}(3,7)$ modulo Sym_7.

One of these cells is not a facet because it lies in the unique cell of dimension six. This means that $\mathrm{Dr}(3,7)$ has $93+1=94$ facets (= maximal cells) up to the Sym_7-symmetry.

Each point in $\mathrm{Dr}(3,n)$ determines a plane in $\mathbb{TP}^{n-1}$. This map was described in [**83, 84**] and we recall it in Section 7.5. The cells of $\mathrm{Dr}(3,n)$ modulo Sym_n correspond to combinatorial types of tropical planes. Facets of $\mathrm{Dr}(3,n)$ correspond to *generic planes* in $\mathbb{TP}^{n-1}$:

COROLLARY 7.3. *The number of combinatorial types of generic planes in* $\mathbb{TP}^6$ *is* 94. *The numbers of types of generic planes in* $\mathbb{TP}^3$, $\mathbb{TP}^4$, *and* $\mathbb{TP}^5$ *are one, one, and seven, respectively.*

PROOF. The unique generic plane in $\mathbb{TP}^3$ is the cone over the complete graph K_4. Planes in $\mathbb{TP}^4$ are parameterized by the Petersen graph $\mathrm{Dr}(3,5)=\mathrm{Gr}(3,5)$, and the unique generic type is dual to the trivalent tree with five leaves. The seven types of generic planes in $\mathbb{TP}^5$ were derived in [**84**, Section 5]. Drawings of their bounded parts are given in Figure 7.1, while their unbounded cells are represented by the tree arrangements in Table 2 below. The number 94 for $n=7$ is derived from Theorem 7.2. □

A complete census of all combinatorial types of tropical planes in $\mathbb{TP}^6$ is posted at

`www.uni-math.gwdg.de/jensen/Research/G3_7/grassmann3_7.html`.

This web site and the notation used therein is a main contribution of this chapter. In the rest of this section we explain how our two classification theorems were obtained.

COMPUTATIONAL PROOF OF THEOREM 7.1. The Grassmannian $\mathrm{Gr}(3,7)$ is the tropical variety defined by the Plücker ideal $I_{3,7}$ in the polynomial ring $\mathbb{K}[p_S]$ in 35 unknowns. We first suppose that $\mathbb{K}$ has characteristic zero, and for our computations we take $\mathbb{K}=\mathbb{Q}$. The subvariety of $\mathbb{P}^{34}_{\mathbb{Q}}$ defined by $I_{3,7}$ is irreducible of dimension 12 and has an effective six-dimensional torus action. The Bieri-Groves Theorem [**5**] ensures that $\mathrm{Gr}(3,7)$ is a pure five-dimensional subcomplex of the Gröbner complex of $I_{3,7}$. Moreover, by [**13**, Theorem 3.1], this complex is connected in codimension one. The software `Gfan` [**56**] exploits this connectivity by traversing the facets exhaustively when computing $\mathrm{Gr}(3,7)=\mathbb{T}(I_{3,7})$.

The input to `Gfan` is a single maximal Gröbner cone of the tropical variety. The cone is, as described in the `Gfan` manual, represented by a pair of Gröbner bases. Knowing a relative interior point of a maximal cone we can compute this pair with the command

```
gfan_initialforms --ideal --pair
```

run on the input

```
Q[p123,p124,p125,p126,p127,p134,p135,p136,p137,p145,p146,p147,
p156,p157,p167,p234,p235,p236,p237,p245,p246,p247,p256,p257,p267,
p345,p346,p347,p356,p357,p367,p456,p457,p467,p567]
{
p123*p145-p124*p135+p125*p134,
....
p123*p456-p124*p356+p125*p346+p126*p345,
....
```

```
p347*p567-p357*p467+p367*p457
}
( 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, -3, -2, -2, -3, -2, 0, 0, 0, 0,
-3, -1, -2, -1, -2, -1, -2, -1, -3, -1, -2, -1, -3, -4, -3, -5).
```

The polynomials are the 140 quadrics which minimally generate the Plücker ideal $I_{3,7}$. Among these are 105 three-term relations and 35 four-term relations. Since `Gfan` uses the max-convention for tropical addition, weight vectors have to be negated. The output is handed over to the program `gfan_tropicaltraverse`, which computes all other maximal cones. For this computation to finish it is decisive to use the `-symmetry` option. The symmetric group Sym_7 acts on the tropical Plücker coordinates as a subgroup of Sym_{35}. In terms of classical Plücker coordinates, these symmetries only exist if we simultaneously perform sign changes, such as $p_{132} = -p_{123}$. We inform `Gfan` about these sign changes using `-torus`, and we specify the sign changes on the input as elements of $\{-1, +1\}^{35}$ together with the generators of $\mathrm{Sym}_7 \subset \mathrm{Sym}_{35}$ after the Gröbner basis pair produced above:

```
{(15,16,17,18,0,19,20,21,1,22,23,2,24,3,4,25,26,27,5,28,29,6,30,7,8,31,
32,9,33,10,11,34,12,13,14),(0,1,2,3,4,15,16,17,18,19,20,21,22,23,24,5,
6,7,8,9,10,11,12,13,14,25,26,27,28,29,30,31,32,33,34)}
{(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),
(-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1)}
```

Before traversing $\mathrm{Gr}(3, 7)$, `Gfan` verifies algebraically that these indeed are symmetries.

In order to handle a tropical variety as large as $\mathrm{Gr}(3, 7)$, the implementation of the traversal algorithm in [**13**] was improved in several ways. During the traversal of the maximal cones up to symmetry, algebraic tests were translated into polyhedral containment questions whenever possible. Since the fan turned out to be simplicial, computing the rays could be reduced to linear algebra while in general `Gfan` uses the double description method of `cddlib` [**36**]. In the subsequent combinatorial extraction of all faces up to symmetry, checking if two cones are in the same orbit can be done at the level of canonical interior points. Checking if two points are equal up to symmetry was done by running through all permutations in the group. This may not be optimal but is sufficient for our purpose. For further speed-ups we linked `Gfan` to the floating point LP solver `SoPlex` [**96**] which produced certificates verifiable in integer arithmetic. In case of a failure caused by round-off errors, the program falls back on `cddlib` which solves the LP problem in exact arithmetic. The running time for the computation is approximately 25 hours on a standard desktop computer with `Gfan` version 0.4, which is expected to be released in the fall of 2008. The output of `Gfan` is in `polymake` [**37**] format, and the program `homology` [**31**] was used to compute the integral homology.

The above computations established our result in characteristic zero. To obtain the same result for prime characteristics $p \geq 3$, we used `Macaulay2` to redo all Gröbner basis computations, one for each cone in $\mathrm{Gr}(3, 7)$, in the polynomial ring $\mathbb{Z}[p_S]$ over the integers. We found that all but one of the initial ideals $\mathrm{in}_\lambda(I_{3,6})$ arise from $I_{3,6}$ via a Gröbner basis whose coefficients are $+1$ and -1. Hence these cones of $\mathrm{Gr}(3, 7)$ are characteristic-free. The only exception is the Fano cone which will be discussed in the end of Section 7.3. □

Computational proof of Theorem 7.2. For $d = 3$ and $n = 7$ there are 105 three-term Plücker relations (7.1). A vector $\lambda \in \mathbb{R}^{35}$ lies in $\mathbf{Dr}(3, 7)$ if and only if the initial form of each three-term relation with respect to λ has either two or three terms. There are four possibilities for this to happen, and each choice is described by a linear system of equations and inequalities. This system is feasible if and only if the corresponding cone exists in the Dressian $\mathbf{Dr}(3, 7)$, which can be tested using linear programming. In theory, we could compute the Dressian by running a loop over all 4^{105} choices and list which choices determine a non-empty cone of $\mathbf{Dr}(3, 7)$. Clearly, this is infeasible in practice.

To control the combinatorial explosion, we employed the representation of tropical planes by abstract tree arrangements which will be introduced in Section 4. This representation allows a recursive computation of $\mathbf{Dr}(3, n)$ from $\mathbf{Dr}(3, n - 1)$. The idea is similar to what is described in the previous paragraph, but the approach is much more efficient. By taking the action of the symmetric group of degree n into account and by organizing this exhaustive search well enough, this leads to a viable computation. A key issue seems to be to focus on the equations early in the enumeration, while the inequalities are considered only at the very end. A `polymake` implementation enumerates all cones of $\mathbf{Dr}(3, 7)$ within one hour. The same computation for $\mathbf{Dr}(3, 6)$ takes less than two minutes.

Again we used `homology` for computing the integral homology of $\mathbf{Dr}(3, 7)$. Since the fan $\mathbf{Dr}(3, 7)$ is not simplicial it cannot be fed into `homology` directly. However, it is homotopy equivalent to its crosscut complex, which thus has the same homology [**11**]. The *crosscut complex* (with respect to the atoms) is the abstract simplicial complex whose vertices are the rays of $\mathbf{Dr}(3, 7)$ and whose faces are the subsets of rays which are contained in cones of $\mathbf{Dr}(3, 7)$. The computation of the homology of the crosscut complex takes about two hours. □

Remark 7.4. Following Dress and Wenzel [**29, 30**], a *valuated matroid* of rank d on the set $[n]$ is a map $\pi : [n]^d \to \mathbb{R} \cup \{\infty\}$ such that $\pi(\omega)$ is independent of the ordering of the sequence ω, $\pi(\omega) = \infty$ if an element occurs twice in ω, and the following axiom holds: for every $(d - 1)$-subset σ and every $(d + 1)$-subset $\tau = \{\tau_1, \tau_2, \ldots, \tau_{d+1}\}$ of $[n]$ the minimum of

$$\pi(\sigma \cup \{\tau_i\}) + \pi(\tau \backslash \{\tau_i\}) \quad \text{for} \quad 1 \le i \le d + 1$$

is attained at least twice. Results of Dress and Wenzel [**29**] imply that tropical Plücker vectors and valuated matroids are the same. To see this, one applies [**29**, Theorem 3.4] to the perfect fuzzy ring arising from $(\mathbb{R} \cup \{\infty\}, \min, +)$ via the construction described in [**29**, page 182].

7.3. Matroid Subdivisions

A *weight function* λ on an n-dimensional polytope P in $\mathbb{R}^n$ assigns a real number to each vertex of P. The lower facets of the lifted polytope $\operatorname{conv}\{(v, \lambda(v)) \mid v \text{ vertex of } P\}$ in $\mathbb{R}^{n+1}$ induce a polytopal subdivision of P. Polytopal subdivisions arising in this way are called *regular*. The set of all weights inducing a fixed subdivision forms a (relatively open) polyhedral cone, and the set of all these cones is a complete fan, the *secondary fan* of P. The dimension of the secondary fan as a spherical complex is $m-n-1$, where m is the number of vertices of P. For a detailed introduction to these concepts see [**19**].

We denote the canonical basis vectors of $\mathbb{R}^n$ by $e_1, e_2, \ldots, e_n$, and we abbreviate $e_X := \sum_{i \in X} e_i$ for any subset $X \subseteq [n]$. For a set $\mathcal{X} \subseteq \binom{[n]}{d}$ we define the polytope

$$P_{\mathcal{X}} \ := \ \operatorname{conv}\{e_X \mid X \in \mathcal{X}\} \ .$$

The d-th *hypersimplex* in $\mathbb{R}^n$ is the special case

$$\Delta(d,n) \ := \ P_{\binom{[n]}{d}} \, .$$

A subset $\mathcal{M} \subseteq \binom{[n]}{d}$ is a *matroid* of *rank* d on the set $[n]$ if the edges of the polytope $P_{\mathcal{M}}$ are all parallel to the edges of $\Delta(d,n)$; in this case, $P_{\mathcal{M}}$ is called a *matroid polytope*, and the elements of $\mathcal{M}$ are the *bases*. That this definition really describes a matroid as, for example, in White [**94**], is a result of Gel′fand, Goresky, MacPherson, and Serganova [**38**]. Moreover, each face of a matroid polytope is again a matroid polytope [**35**]. A polytopal subdivision of $\Delta(d,n)$ is a *matroid subdivision* if each of its cells is a matroid polytope.

Proposition 7.5 (Speyer [**83**], Proposition 2.2). *A weight vector* $\lambda \in \mathbb{R}^{\binom{[n]}{d}}$ *lies in the Dressian* $\operatorname{Dr}(d,n)$ *if and only if it induces a matroid subdivision of* $\Delta(d,n)$.

The weight functions inducing matroid subdivisions form a subfan of the secondary fan of $\Delta(d,n)$, and this defines the *secondary fan structure* on the Dressian $\operatorname{Dr}(d,n)$. It is not obvious whether the secondary fan structure and the Plücker fan structure on $\operatorname{Dr}(d,n)$ coincide. We shall see in Theorem 7.15 that this is indeed the case if $d=3$. In particular, the rays of the Dressian $\operatorname{Dr}(3,n)$ correspond to coarsest matroid subdivisions of $\Delta(3,n)$.

Corollary 7.6. *Let* M *be a connected matroid of rank* d *on* $[n]$ *and let* $\lambda_M \in \{0,1\}^{\binom{[n]}{d}}$ *be the vector which satisfies* $\lambda_M(X) = 0$ *if* X *is a basis of* M *and* $\lambda_M(X) = 1$ *if* X *is not a basis of* M. *Then* λ_M *lies in the Dressian* $\operatorname{Dr}(d,n)$, *and the corresponding matroid decomposition of* $\Delta(d,n)$ *has the matroid polytope* P_M *as a maximal cell.*

Proof. The basis exchange axiom for matroids translates into a combinatorial version of the quadratic Plücker relations (cf. Remark 7.4), and this ensures that the vector λ_M lies in the Dressian $\operatorname{Dr}(d,n)$. By Proposition 7.5, the regular subdivision of $\Delta(d,n)$ defined by λ_M is a matroid subdivision. The matroid polytope P_M appears as a lower face in the lifting of $\Delta(d,n)$ by λ_M, and hence it is a cell of the matroid subdivision. It is a maximal cell because $\dim(P_M) = d-1$ if and only if the matroid M is connected; see [**35**]. □

Each vertex figure of $\Delta(d,n)$ is isomorphic to the product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$; see Example 3.4 (a). A regular subdivision of a polytope induces regular subdivisions on its facets as well as on its vertex figures. For hypersimplices the converse holds (see also Proposition 7.16):

PROPOSITION 7.7 (Kapranov [**60**], Corollary 1.4.14). *Each regular subdivision of the product of simplices* $\Delta_{d-1} \times \Delta_{n-d-1}$ *is induced by a regular matroid subdivision of* $\Delta(d,n)$.

A *split* of a polytope is a regular subdivision with exactly two maximal cells. Lemma 2.54 states that every split of $\Delta(d,n)$ is a matroid subdivision. Collections of splits that are pairwise compatible define a simplicial complex, known as the *split complex* of $\Delta(d,n)$. It was shown in Section 2.7 that the regular subdivision defined by pairwise compatible splits is always a matroid subdivision. The following result is a reformulation of Proposition 2.61:

PROPOSITION 7.8. *The split complex of* $\Delta(d,n)$ *is a simplicial subcomplex of the Dressian* $\mathrm{Dr}(d,n)$, *with its secondary complex structure. They are equal if* $d = 2$ *(or if* $d = n-2$*).*

Special examples of splits come about in the following way. The vertices adjacent to a fixed vertex of $\Delta(d,n)$ span a hyperplane which defines a split; and these splits are called *vertex splits*. Moreover, two vertex splits are compatible if and only if the corresponding vertices of $\Delta(d,n)$ are not connected by an edge. Hence the simplicial complex of stable sets of the edge graph of $\Delta(d,n)$ is contained in the split complex of $\Delta(d,n)$.

COROLLARY 7.9. *The simplicial complex of stable sets of the edge graph of the hypersimplex* $\Delta(d,n)$ *is a subcomplex of* $\mathrm{Dr}(d,n)$. *Hence, the dimension of the Dressian* $\mathrm{Dr}(d,n)$ *is bounded below by one less than the maximal size of a stable set of this edge graph.*

We shall use this corollary to prove the main result in this section. Recall that the dimension of the Grassmannian $\mathrm{Gr}(3,n)$ equals $2n-9$. Consequently, the following theorem implies that, for large n, most of the tropical planes (cf. Section 7.5) are not realizable.

THEOREM 7.10. *The dimension of the Dressian* $\mathrm{Dr}(3,n)$ *is of order* $\Theta(n^2)$.

For the proof of this result we need one more definition. The *spread* of a vector in $\mathrm{Dr}(d,n)$ is the number of maximal cells of the corresponding matroid decomposition. The splits are precisely the vectors of spread two, and these are rays of $\mathrm{Dr}(d,n)$. The rays of $\mathrm{Dr}(3,6)$ are either of spread two or three; see [**84**, Section 5]. As a result of our computation the spreads of rays of $\mathrm{Dr}(3,7)$ turn out to be two, three, and four. We note the following result.

PROPOSITION 7.11. *As* n *increases, the spread of the rays of* $\mathrm{Dr}(3,n)$ *is not bounded by a constant.*

PROOF. By Proposition 7.7, each regular subdivision of $\Delta_2 \times \Delta_{n-4}$ is induced by a regular matroid subdivision of $\Delta(3,n)$, and hence, in light of the Cayley trick [**80**], by mixed subdivisions of the dilated triangle $(n-3)\Delta_2$; see also Section 7.4. This correspondence maps rays of the secondary fan of $\Delta_2 \times \Delta_{n-4}$ to rays of $\mathrm{Dr}(3, n)$. Now, a coarsest mixed subdivision of $(n-3)\Delta_2$ can have arbitrarily many polygons as n grows large. For an example consider the hexagonal subdivision in [**80**, Figure 12]. Hence a coarsest regular matroid subdivision of $\Delta(3, n)$ can have arbitrarily many facets. □

Proof of Theorem 7.10. Speyer [**83**, Theorem 6.1] showed that the spread of any vector in $\mathrm{Dr}(d, n)$ is at most $\binom{n-2}{d-1}$. This is the maximal number of facets of any matroid subdivision of $\Delta(d, n)$. Consider a flag of faces $F_1 \subset F_2 \subset \cdots$ in $\mathrm{Dr}(d, n)$. For every i the subdivision corresponding to F_i has more facets than that of F_{i-1}. Hence $\binom{n-2}{d-1} - 1$ is an upper bound for the dimension of $\mathrm{Dr}(k, n)$. Specializing to $d = 3$, this upper bound is quadratic.

We shall now apply Proposition 7.8 to derive the lower bound. The *generalized Fano matroid* $\mathcal{F}_r$ is a connected simple matroid on $2^r - 1$ points which has rank 3 and is defined as follows. Its three-element circuits are the lines of the $(r-1)$-dimensional projective space $\mathrm{PG}_{r-1}(2)$ over the field $\mathbb{GF}(2)$ with two elements. The total number of unordered bases of $\mathcal{F}_r$, that is, non-collinear triples of points, equals

$$\beta_r := \frac{1}{6}(2^r - 1)(2^r - 2)(2^r - 4).$$

The number of vertices of $\Delta(3, 2^r - 1)$ which are not bases of $\mathcal{F}_r$ equals

$$\nu_r := \binom{2^r - 1}{3} - \beta_r = \frac{1}{6}(2^r - 1)(2^r - 2)$$

We claim that the non-bases of $\mathcal{F}_r$ form a stable set in the edge graph of $\Delta(3, 2^r - 1)$. Indeed, the non-bases are precisely the collinear triplets of points, that is, the full point rows of the lines in $\mathrm{PG}_{r-1}(2)$. Two distinct point rows of lines in $\mathrm{PG}_{r-1}(2)$ share at most one point, and hence the two corresponding vertices of $\Delta(3, 2^r - 1)$ do not differ by an exchange of two bits, which means that they are not connected by an edge.

The quadratic lower bound is now derived from Proposition 7.8 as follows. For given any n, let r be the unique natural number satisfying $2^r - 1 \le n < 2^{r+1}$. Then the generalized Fano matroid $\mathcal{F}_r$ yields a stable set of size $\nu_r = 1/6(2^r-1)(2^r-2) \ge n^2/24 - n/12$ in the edge graph of $\Delta(3, n)$. The latter inequality follows from $2^r - 1 \ge n/2$. □

Computational proof of Theorem 7.1 (continued). We still have to discuss the Fano cone of $\mathrm{Dr}(3, 7)$ and its relationship to $\mathrm{Gr}(3, 7)$. The matroid $\mathcal{F}_3$ in the proof of Theorem 7.10 corresponds to the Fano plane $\mathrm{PG}_2(2)$, which is shown in Figure 7.2 on the left. We have $\beta_3 = 28$ and $\nu_3 = 7$. Via Corollary 7.6, the Fano matroid $\mathcal{F}_3$ gives rise to a cone in $\mathrm{Dr}(3, 7)$, which we call the *Fano cone*. The corresponding cell of $\mathrm{Dr}(3, 7)$, seen as a polytopal complex, has dimension six. Moreover, all 30 six-dimensional cells of $\mathrm{Dr}(3, 7)$ come from the Fano matroid $\mathcal{F}_3$ by relabeling. They form a single orbit under the Sym_7 action since the automorphism group $\mathrm{GL}_3(2)$ of $\mathcal{F}_3$ has order $168 = 5040/30$. If the field $\mathbb{K}$ considered has characteristic two then the Fano cell of $\mathrm{Dr}(3, 7)$ intersects $\mathrm{Gr}(3, 7)$ in a five-dimensional complex that looks like a tropical hyperplane.

Finally, suppose that the characteristic of $\mathbb{K}$ is different from two. Then the intersection of the Fano cell with $\mathrm{Gr}(3, 7)$ is a five-dimensional simplicial sphere arising from seven copies of the non-Fano matroid; see Figure 7.2 on the right. In this case this also gives us the difference in the homology of $\mathbf{Dr}(3,7)$ and $\mathbf{Gr}(3,7)$. The Fano 6-cells are simplices. Each of them cancels precisely one homology cycle of $\mathbf{Gr}(3,7)$. □

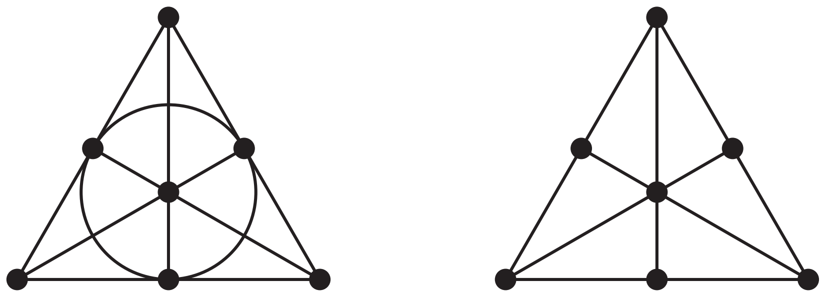

FIGURE 7.2. The point configurations for the Fano and non-Fano matroids.

In spite of the results in this sections, many open problems remain. Here are two specific questions we have concerning the combinatorial structure of the Dressian $\mathbf{Dr}(3,n)$:

- ▹ Are all rays of $\mathbf{Dr}(3,n)$ always rays of $\mathbf{Gr}(3,n)$?
- ▹ Characterize the rays of $\mathbf{Dr}(3,n)$, that is, coarsest matroid subdivisions of $\Delta(3,n)$.

7.4. Tree Arrangements

Let $n \geq 4$ and consider an n-tuple of metric trees $T = (T_1, T_2, \ldots, T_n)$ where T_i has the set of leaves $[n]\backslash\{i\}$. A *metric tree* T_i by definition comes with non-negative edge lengths, and by adding lengths along paths it defines a metric $d_i : ([n]\backslash\{i\}) \times ([n]\backslash\{i\}) \to \mathbb{R}_{\geq 0}$. We call the n-tuple T of metric trees a *metric tree arrangement* if

$$d_i(j,k) \;=\; d_j(k,i) \;=\; d_k(i,j) \tag{7.2}$$

for all $i,j,k \in [n]$ pairwise distinct. Moreover, considering trees T_i without metrics, but with leaves still labeled by $[n]\backslash\{i\}$, we say that T is an *abstract tree arrangement* if

- ▹ either $n = 4$;
- ▹ or $n = 5$, and T is the set of quartets of a tree with five leaves;
- ▹ or $n \geq 6$, and $(T_1\backslash i, \ldots, T_{i-1}\backslash i, T_{i+1}\backslash i, \ldots, T_n\backslash i)$ is an arrangement of $n-1$ trees for each $i \in [n]$.

Here $T_j\backslash i$ denotes the tree on $[n]\backslash\{i,j\}$ gotten by deleting leaf i from tree T_j. A *quartet* of a tree is a subtree induced by four of its leaves.

The following result relates the two concepts of tree arrangements we introduced:

PROPOSITION 7.12. *Each metric tree arrangement gives rise to an abstract tree arrangement by ignoring the edge lengths. The converse is not true: for $n \geq 9$, there exist abstract arrangements of n trees that do not support any metric tree arrangement.*

PROOF. The first assertion follows from the Four Point Condition; see [**73**, Theorem 2.36]. An example establishing the second assertion is the abstract arrangement of nine trees listed in Table 1, depicted in Figure 7.3, and explained in Example 7.18. □

The hypersimplex $\Delta(d,n)$ is the intersection of the unit cube $[0,1]^n$ with the affine hyperplane $\sum x_i = d$. From this it follows that the facets of $\Delta(d,n)$ correspond to the facets of $[0,1]^n$. We call the facet defined by $x_i = 0$ the *i-th deletion facet* of $\Delta(d,n)$, and the facet defined by $x_i = 1$ the *i-th contraction facet*. These names come about as follows: If $\mathcal{M}$ is a matroid on $[n]$ of rank d, then the intersection of $P_{\mathcal{M}}$ with the i-th deletion (contraction) facet is the matroid polytope of the matroid obtained by deleting (contracting) i. Each deletion facet of $\Delta(d,n)$ is isomorphic to $\Delta(d,n-1)$, and each contraction facet is isomorphic to $\Delta(d-1,n-1)$. We use the terms "deletion" and "contraction" also for matroid subdivisions and for vectors in $\mathbb{R}^{\binom{[n]}{d}}$.

LEMMA 7.13. *Each matroid subdivision Σ of $\Delta(3,n)$ defines an abstract arrangement $T(\Sigma)$ of n trees. Moreover, if Σ is regular then $T(\Sigma)$ supports a metric tree arrangement.*

PROOF. Each contraction facet of $\Delta(3,n)$ is isomorphic to $\Delta(2,n-1)$, and Σ induces matroid subdivisions on n copies of $\Delta(2,n-1)$. But the matroid subdivisions of $\Delta(2,n-1)$ are generated by compatible systems of splits and thus are dual to trees. Hence Σ gives rise to a tree arrangement.

Now let Σ be regular with weight function λ. By adding or subtracting a suitable multiple of $(1,1,\ldots,1)$ and subsequent rescaling, we can assume that λ attains values between 1 and 2 only. The induced regular subdivisions of $\Delta(2,n-1)$ are dual to trees with $n-1$ leaves. A weight function on $\Delta(2,n-1)$ which takes values between 1 and 2

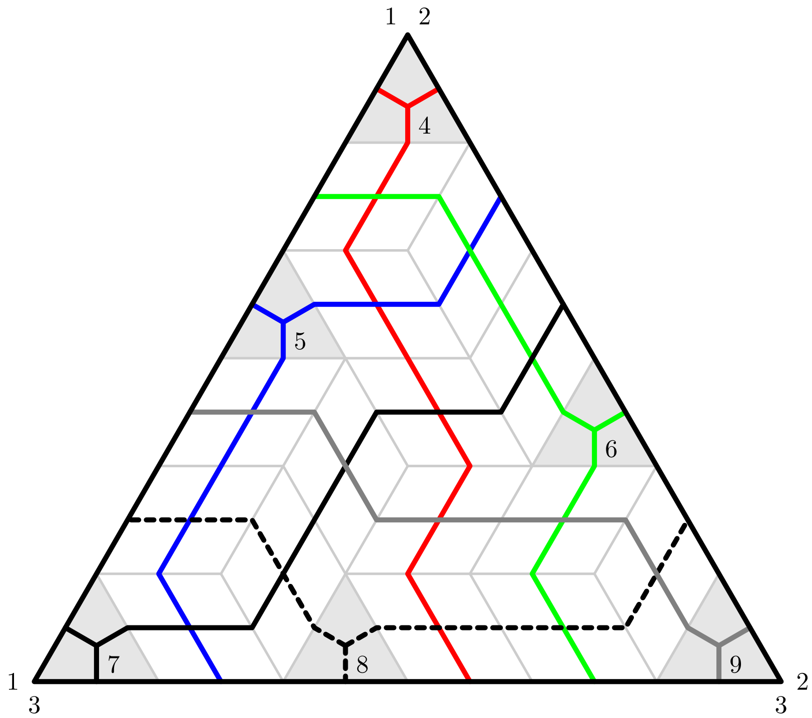

FIGURE 7.3. Abstract arrangement of nine caterpillar trees on eight leaves encoding a matroid subdivision of $\Delta(3,9)$ which is not regular; see Table 1.

is a metric. The Split Decomposition Theorem of Bandelt and Dress [**4**, Theorem 2] allows to read off the lengths on all edges of these trees; see also Theorem 2.21. □

PROPOSITION 7.14. *Let* Σ *and* $\bar{\Sigma}$ *be two matroid subdivisions of* $\Delta(3,n)$ *such that* Σ *refines* $\bar{\Sigma}$. *If* Σ *and* $\bar{\Sigma}$ *induce the same subdivision on the boundary of* $\Delta(3,n)$ *then* Σ *and* $\bar{\Sigma}$ *coincide.*

PROOF. Suppose that Σ strictly refines $\bar{\Sigma}$. Then there is a codimension-one-cell F of Σ which is not a cell in $\bar{\Sigma}$. Let $\bar{F}$ be the unique full-dimensional cell of $\bar{\Sigma}$ that contains F. In particular, F is not contained in the boundary of $\Delta(3,n)$. Then F is a rank-three-matroid polytope $F = P_{\mathcal{M}}$ of codimension one where $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ is the disjoint union of a rank-one-matroid $\mathcal{M}_1$ and a rank-two-matroid $\mathcal{M}_2$. In particular, $F \cong P_{\mathcal{M}_1} \times P_{\mathcal{M}_2}$. Notice that the affine hull H of F is defined by the equation $\sum_{i \in I} x_i = 1$ where we denote by I the set of elements of $\mathcal{M}_1$, all of which are parallel because of $\operatorname{rank} \mathcal{M}_1 = 1$.

Since $\bar{F}$ is subdivided by H there exist vertices v, w of $\bar{F}$ on either side of H. Now $\bar{F}$ is also a matroid polytope of some matroid $\bar{\mathcal{M}}$ containing $\mathcal{M}$ as a submatroid. Up to relabeling we can assume that $v = e_{12i}$ and $w = e_{345}$ such that $\{1,2,i\}$ and $\{3,4,5\}$ are bases of $\bar{\mathcal{M}}$ which are not bases of $\mathcal{M}$ and where $1,2 \in I$ and $i,3,4,5 \notin I$. If $i \notin \{3,4,5\}$ we can exchange i in the basis $\{1,2,i\}$ by some $j \in \{3,4,5\}$ to obtain a new basis of $\bar{\mathcal{M}}$. Without loss of generality we can assume that $i = 5$ or $j = 5$. Hence $\{1,2,5\}$ and $\{3,4,5\}$ are bases of $\bar{\mathcal{M}}$ that are not bases of $\mathcal{M}$. Notice that e_{125} and e_{345} are still on different sides of H as e_{12i} and e_{125} are connected by an edge and $\{1,2,5\}$ is not a basis of $\mathcal{M}$.

Now, as $\operatorname{rank} \mathcal{M}_i \leq 2$, both $\mathcal{M}_1$ and $\mathcal{M}_2$ are realizable as affine point configurations over $\mathbb{R}$, and hence so is $\mathcal{M}$. In the sequel we identify $\mathcal{M}$ with a suitable point configuration (with multiple points). This way we obtain a subconfiguration of five points in $\mathcal{M}$ which looks like one of the two configurations shown in Figure 7.4.

FIGURE 7.4. Two point configurations in the Euclidean plane.

Consider the intersection of $\Delta(3,n)$ with the affine space defined by $x_5 = 1$ and $x_6 = x_7 = \cdots = x_n = 0$. This gives us an octahedron $C \cong \Delta(2,4)$ in the boundary of $\Delta(3,n)$. The intersection $S = F \cap C$ is a square; it can be read off Figure 7.4 as the convex hull of the four points e_{135}, e_{145}, e_{235}, and e_{245}. In particular, the square S is a cell of Σ. However, since e_{125} and e_{345} are vertices of $\bar{F} = P_{\bar{\mathcal{M}}}$ as discussed above, C is a cell of $\bar{\Sigma}$. We conclude that the square S is a cell of Σ but not a cell of $\bar{\Sigma}$. By construction, $S \subset C$ is contained in the boundary of $\Delta(3,n)$. This yields the desired contradiction, as Σ and $\bar{\Sigma}$ induce the same subdivision on the boundary. □

Two metric tree arrangements are *equivalent* if they induce the same abstract tree arrangement. The following is the main result of this section.

THEOREM 7.15. *The equivalence classes of arrangements of n metric trees are in bijection with the regular matroid subdivisions of the hypersimplex $\Delta(3,n)$. Moreover, the secondary fan structure on $\mathrm{Dr}(3,n)$ coincides with the Plücker fan structure.*

PROOF. By Lemma 7.13, each regular matroid subdivision defines a metric tree arrangement. The harder direction is to show that each metric tree arrangement gives rise to a regular matroid subdivision. We will prove this by induction on n. The hypersimplex $\Delta(3,4)$ is a 3-simplex without any non-trivial subdivisions, and $\mathrm{Dr}(3,4)$ is a single point corresponding to the unique equivalence class of metric trees. The hypersimplex $\Delta(3,5)$ is isomorphic to $\Delta(2,5)$, and $\mathrm{Dr}(3,5) = \mathrm{Gr}(3,5) \cong \mathrm{Gr}(2,5)$ is isomorphic to the Petersen graph (considered as a one-dimensional polytopal complex). Also in this case the result can be verified directly. This establishes the basis of our induction, and we now assume $n \geq 6$.

Let T be an arrangement of n metric trees with tree metrics $d_1, d_2, \ldots, d_n$. In view of the axiom (7.2), the following map $\pi : [n]^3 \to \mathbb{R} \cup \{\infty\}$ is well-defined:

$$\pi(i,j,k) \;=\; \begin{cases} d_i(j,k) = d_j(k,i) = d_k(i,j) & \text{if } i,j,k \text{ are pairwise distinct,} \\ \infty & \text{otherwise.} \end{cases}$$

In order to show that π is a tropical Plücker vector we have to verify that the minimum

$$\min\{\pi_{hij} + \pi_{hkl},\ \pi_{hik} + \pi_{hjl},\ \pi_{hil} + \pi_{hjk}\}$$

is attained at least twice, for any pairwise distinct $h,i,j,k,l \in [n]$. Now, since $n \geq 6$, each 5-tuple in $[n]$ is already contained in some deletion, and hence the desired property is satisfied by induction. We conclude that the restriction of the map π to increasing triples $i < j < k$ is a finite tropical Plücker vector, that is, it is an element of $\mathrm{Dr}(3,n)$. By Proposition 7.5, the map π defines a matroid subdivision $\Sigma(T)$ of $\Delta(3,n)$.

Consider any metric tree arrangement T' that is equivalent to T. The maps π and π' associated with T and T' respectively clearly lie in the same cone of the Plücker fan structure on $\mathrm{Dr}(3,n)$. What we must prove is that they are also in the same cone of the secondary fan structure on $\mathrm{Dr}(3,n)$. Equivalently, we must show that $\Sigma(T') = \Sigma(T)$.

Suppose the secondary fan structure on $\mathrm{Dr}(3,n)$ is strictly finer than the Plücker fan structure. Then there is a regular matroid subdivision Σ of $\Delta(3,n)$ whose secondary cone $S(\Sigma)$ is strictly contained in the corresponding cone $P(\Sigma)$ of tropical Plücker vectors. We fix a weight function λ in the boundary of $S(\Sigma)$ which is contained in the interior of $P(\Sigma)$. The matroid subdivision induced by λ is denoted by $\bar{\Sigma}$. By construction, Σ strictly refines $\bar{\Sigma}$, and, by induction, we can assume that Σ and $\bar{\Sigma}$ induce the same subdivision on the entire boundary of $\Delta(3,n)$. Due to Proposition 7.14, we have that $\Sigma = \bar{\Sigma}$, and this completes our proof. ◻

We saw in Proposition 7.7 that each regular subdivision of $\Delta_2 \times \Delta_{n-4}$ is induced by a regular matroid subdivision of $\Delta(3,n)$. This implies that $\mathrm{Dr}(3,n)$ contains a distinguished $(2n-9)$-dimensional sphere, dual to the secondary polytope of $\Delta_2 \times \Delta_{n-4}$, which parameterizes all arrangements of $n-3$ lines in the tropical plane $\mathbb{TP}^2$. It has the following nice description in terms of tree arrangements. Let $L_1, L_2, \ldots, L_{n-3}$ be the $n-3$ lines and let L_x, L_y, L_z denote the three boundary lines of $\mathbb{TP}^2$. Each of these n lines translates into a tree. The tree for L_x is obtained by branching off the leaves $\{1, 2, \ldots, n-3\}$ on the path between leaves y and z, in the order in which the L_j intersect L_x. The trees for L_y and L_z are analogous. The tree for L_i has one distinguished node with long branches to the three special leaves x, y and z. Along the path from the distinguished node to leaf x we branch off additional leaves j for each line L_j that intersects the line L_i in its x-halfray. This branching takes place in the order in which the lines L_j intersect L_j. In this manner, every arrangement of $n-3$ lines in $\mathbb{TP}^2$ translates into an arrangement of n trees.

The same construction also applies to arrangements of $n-3$ tropical pseudolines in $\mathbb{TP}^2$ as defined by Ardila and Develin [**2**]. We shall now describe this in terms of lozenge tilings as in [**80**]. Let Σ be any polytopal subdivision of $\Delta_2 \times \Delta_{n-4}$. The Cayley Trick encodes Σ as a mixed subdivision $M(\Sigma)$ of $(n-3)\Delta_2$, a regular triangle of side length $n-3$. By [**80**, Theorem 3.5] the mixed subdivisions of dilated triangles are characterized as those polygonal subdivisions whose cells are tiled by lozenges and upward triangles (with unit edge lengths). Here a *lozenge* is a parallelogram which is the union of one upward triangle and one downward triangle. We call a mixed cell *even* if it can be tiled by lozenges only. Those which need an upward triangle in any tiling are *odd*. A counting argument now reveals that each mixed subdivision of $(n-3)\Delta_2$ contains up to $n-3$ odd polygonal cells.

PROPOSITION 7.16. *Each polytopal subdivision Σ of $\Delta_2 \times \Delta_{n-4}$, or each mixed subdivision $M(\Sigma)$ of the triangle $(n-3)\Delta_2$, determines an abstract arrangement $T(\Sigma)$ of n trees.*

PROOF. Assume that Σ is a triangulation. Equivalently, $M(\Sigma)$ has exactly $n-3$ odd cells, all of which are upward triangles, and the even cells are lozenges. Placing a labeled node into each upward triangle defines a tree in the dual graph of Σ. Each of its three branches consists of the edges in $M(\Sigma)$ which are in the same parallel class as one fixed edge of that upward triangle. Two opposite edges in a lozenge are parallel, and the *parallelism* that we refer to is the transitive closure of this relation. Each parallel class of edges extends to the boundary of the triangle $(n-3)\Delta_2$. Doing so for all the upward triangles, we obtain an arrangement of tropical pseudo-lines [**2**]. Each of these is subdivided by the intersection with the other tropical pseudo-lines. We further add the three boundary lines of the big triangle to the arrangement. This specifies an abstract tree arrangement $T(\Sigma)$. Note that the trees in the arrangement partition the dual graph of the lozenge tiling.

Now we consider the situation where Σ is not a triangulation, so $M(\Sigma)$ is a coarser mixed subdivision of $(n-3)\Delta_2$. We shall associate a tree arrangement with $M(\Sigma)$. Pick any triangulation Σ' which refines Σ. Then by the above procedure we have an abstract tree arrangement $T(\Sigma')$ induced by Σ'. Then, as Σ' refines Σ, one can contract edges in the trees of the arrangement $T(\Sigma')$. In this way, one also arrives at an abstract arrangement of n trees. Three of them correspond to the boundary lines of $(n-3)\Delta_2$. The $n-3$ non-boundary trees are assigned to the $\leq n-3$ odd cells. Each cell is assigned at least one tree. We note that $T(\Sigma)$ might depend on the choice of the triangulation Σ'. ◻

EXAMPLE 7.17. Let $n = 6$ and consider the two mixed subdivisions of $3\Delta_2$ shown in Figure 7.5. The left one is a lozenge tiling which encodes a regular triangulation of $\Delta_2 \times \Delta_2$, here regarded as the vertex figure of $\Delta(3, 6)$ at e_{135}. There are precisely three

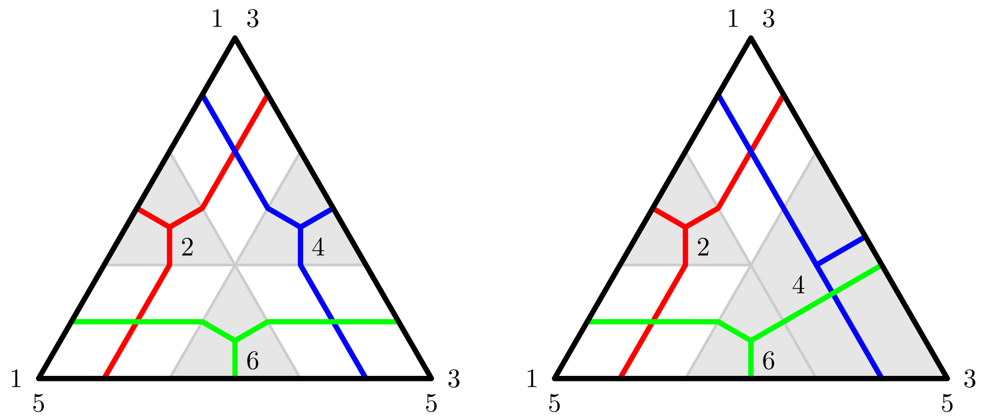

FIGURE 7.5. Mixed subdivisions of $3\Delta_2$ and abstract arrangements of six trees.

upward triangles, and each of them corresponds to a tree. Moreover, the three sides of the big triangle encode three more trees. Using the notation of Figure 7.6, this abstract tree arrangement equals

$$34\,2\,56\,,\;\; 34\,1\,56\,,\;\; 12\,4\,56\,,\;\; 12\,3\,56\,,\;\; 12\,6\,34\,,\;\; 12\,5\,34\,. \tag{7.3}$$

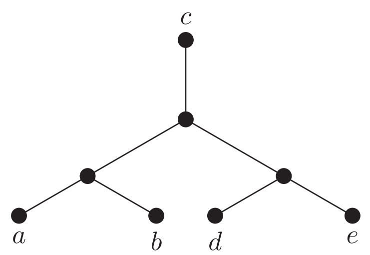

FIGURE 7.6. We use the notation $ab\,c\,de$ for this tree on five labeled leaves.

The tiling of $3\Delta_2$ on the right in Figure 7.5 is a mixed subdivision which coarsens the lozenge tiling discussed above. It corresponds to the abstract tree arrangement

$$34\,2\,56\,,\;\; 34\,1\,56\,,\;\; 12\,(456)\,,\;\; 12\,(356)\,,\;\; 12\,6\,34\,,\;\; 12\,5\,34\,.$$

The tree $ab\,(cde)$ is obtained from the tree $ab\,c\,de$ by contracting the interior edge between c and the pair de. The odd polygonal cells (shaded in Figure 7.5) correspond to trees. □

EXAMPLE 7.18. An example of a non-regular matroid subdivision arises from the lozenge tiling of $6\Delta_2$ borrowed from [**80**] and shown in Figure 7.3. This picture translates into the abstract arrangement of nine trees in Table 1. The corresponding matroid subdivision of $\Delta(3,9)$ is not regular. The Dressian $\mathrm{Dr}(3,9)$ has no cell for this tree arrangement. □

REMARK 7.19. There are 187 lozenge tilings of $4\Delta_2$, each representing 24 triangulations of $\Delta_3 \times \Delta_2$ via the $4! = 24$ ways of labeling the upward triangles. Each lozenge tiling defines an arrangement of seven trees that indexes a maximal cell of $\mathrm{Dr}(3,7)$.

TABLE 1. Abstract arrangement of nine caterpillar trees on eight leaves encoding a matroid subdivision of $\Delta(3,9)$ which is not regular; see Figure 7.3. The notation for caterpillar trees is explained in Figure 7.8 below.

Tree 1: C(24, 6598, 37)	Tree 2: C(14, 5768, 39)	Tree 3: C(17, 5846, 29)
Tree 4: C(12, 6579, 38)	Tree 5: C(26, 4198, 37)	Tree 6: C(14, 5729, 38)
Tree 7: C(13, 5894, 26)	Tree 8: C(15, 7346, 29)	Tree 9: C(15, 7468, 23)

In other words, the polytopal 5-sphere dual to the secondary polytope of $\Delta_2 \times \Delta_3$ has $4488 = 187 \cdot 24$ facets, and embeds as a subcomplex into $\mathrm{Dr}(3,7)$. It is instructive to study this subcomplex by browsing our website for $\mathrm{Dr}(3,7)$. For example, the tropical plane of type 89 on our website corresponds to Figure 4 in [**2**].

REMARK 7.20. Another important sphere sitting inside the Grassmannian $\mathrm{Gr}(d,n)$, and hence in the Dressian $\mathrm{Dr}(d,n)$, is the *positive Grassmannian* $\mathrm{Gr}^+(d,n)$, due to Speyer and Williams [**85**]. A natural next step would be to introduce and study the *positive Dressian* $\mathrm{Dr}^+(d,n)$. Generalizing [**85**, Section 5], the positive Dressian $\mathrm{Dr}^+(3,n)$ would parameterize *metric arrangements of planar trees*. This space contains the $(2n-9)$-dimensional sphere $\mathrm{Gr}^+(3,n)$. It would be interesting to know whether this inclusion is a homotopy equivalence, to explore relations with cluster algebras, and to extend the computation of $\mathrm{Gr}^+(3,7)$ presented in [**85**]. Incidentally, there is a misprint in the right part of [**85**, Table 2]: the eleventh inequality should be "$-x_5 \leq -14$" instead of "-17". With this correction, we independently verified the f-vector and the rays of its normal fan $F_{3,7}$. □

7.5. Tropical Planes

We are now finally prepared to answer the question raised in the title of this paper. Tropical planes are contractible polyhedral surfaces that are dual to the regular matroid subdivisions of $\Delta(3,n)$. Consider any vector p in $\mathbb{R}^{\binom{n}{3}}$ that lies in the Dressian $\mathrm{Dr}(3,n)$. The associated tropical plane L_p in $\mathbb{TP}^{n-1}$ is the intersection of the tropical hyperplanes

$$\mathbb{T}(p_{ijk}x_l + p_{ijl}x_k + p_{ikl}x_j + p_{jkl}x_i)$$

as $\{i,j,k,l\}$ ranges over all 4-element subsets of $[n]$. By a *tropical plane*, we mean any subset of $\mathbb{TP}^{n-1}$ which has the form L_p for some $p \in \mathrm{Dr}(3,n)$. The tropical plane L_p is realizable as the tropicalization of a classical plane in $\mathbb{P}^{n-1}_{\mathbb{K}}$ if and only if $p \in \mathrm{Gr}(3,n)$. The plane L_p is called *series-parallel* if each cell in the corresponding matroid subdivision of $\Delta(3,n)$ is the graphic matroid of a series-parallel graph. Results of Speyer [**83, 82**] imply:

PROPOSITION 7.21. *Let L be a tropical plane in $\mathbb{TP}^{n-1}$ with $f_0(L)$ vertices, $f_1^b(L)$ bounded edges, $f_1^u(L)$ unbounded edges, $f_2^b(L)$ bounded 2-cells and $f_2^u(L)$ unbounded 2-cells. Then*

$$f_0(L) \leq (n-2)(n-3)/2, \quad f_1^b(L) \leq (n-4)(n-3), \quad f_1^u(L) \leq n(n-3),$$
$$f_2^b(L) \leq (n-4)(n-5)/2, \quad f_2^u(L) \leq 3n(n-1)/2\,.$$

These five inequalities are equalities if and only if L is a series-parallel plane.

The unbounded edges and 2-cells of a tropical plane correspond to the nodes and edges of the n trees in the corresponding tree arrangement. Suppose the trees are trivalent. Then each tree has $n-1$ leaves and $n-3$ nodes, for a total of $f_1^u(L) = n(n-3)$ nodes. Moreover, each tree has $n-4$ interior edges and $n-1$ pendent edges. The latter are double-counted. This explains the number $f_2^u(L) = n(n-4) + n(n-1)/2$ of edges in the tree arrangement representing L. To understand this situation geometrically, we identify $\mathbb{TP}^{n-1}$ with an $(n-1)$-simplex, and we note that the tree arrangement is obtained geometrically as the intersection $L \cap \partial\mathbb{TP}^{n-1}$ of L with the boundary of that simplex.

The **first answer** to our question of how to draw a tropical plane is given by Theorem 7.15: simply **draw the corresponding tree arrangement**. This answer has the following interpretation as an algorithm for enumerating all tropical planes. To draw all (generic) planes L in $\mathbb{TP}^{n-1}$, we first list all trees on $n-1$ labeled leaves. Each labeled tree occurs in n relabelings corresponding to the sets $[n]\backslash\{1\}, [n]\backslash\{2\}, \ldots, [n]\backslash\{n\}$ of labels. Inductively, one enumerates all arrangements of $4, 5, \ldots, n$ trees. This naïve approach works well for $n \leq 6$. The result of the enumeration is that, up to relabeling and restricting to trivalent trees, there are precisely seven abstract tree arrangements for $n = 6$. They are listed in Table 2. Each tree is written as $ab\,c\,de$, the notation introduced in Figure 7.6. We then check that each of the seven abstract tree arrangements supports a metric tree arrangement, and we conclude that $\mathrm{Dr}(3,6)$ has seven maximal cells modulo the natural action of the group Sym_6. The names for the seven types of generic planes are the same as in [**84**, Section 5] and in Figure 7.1.

TABLE 2. The trees corresponding to the seven types of tropical planes in $\mathbb{TP}^5$.

Type	Tree 1	Tree 2	Tree 3	Tree 4	Tree 5	Tree 6	Orbit Size
EEEE	23 6 45	13 5 46	12 4 56	15 3 26	14 2 36	24 1 35	30
EEEG	26 5 34	16 5 34	14 2 56	13 2 56	12 3 46	12 3 45	240
EEFF(a)	25 6 34	15 6 34	12 5 46	12 5 36	12 6 34	12 5 34	90
EEFF(b)	25 6 34	15 6 34	12 6 45	12 6 35	12 6 34	12 5 34	90
EEFG	25 6 34	15 6 34	24 1 56	23 1 56	12 6 34	12 5 34	360
EFFG	34 2 56	34 1 56	12 6 45	12 6 35	12 6 34	12 5 34	180
FFFGG	34 2 56	34 1 56	12 4 56	12 3 56	12 6 34	12 5 34	15

It is easy to translate the seven rows in Table 2 into seven pictures of tree arrangements. For example, the representative for type FFFGG in the last row coincides with (7.3) and its picture appears on the left side in Figure 7.5. It can be checked in the pictures that each of the seven tree arrangements has $f_1^u(L) = 18$ nodes and $f_2^u(L) = 27$ edges.

The **second answer** to our question of how to draw a tropical plane is given by Figure 7.1: simply **draw and label the bounded cells**. The planes L in the last six rows of Table 2 are series-parallel. Here, the complex of bounded cells in L has $f_0(L) = 6$ nodes, $f_1^b(L) = 6$ edges and $f_2^b(L) = 1$ two-dimensional cell. The first type EEEE is not series-parallel: its bounded complex is one-dimensional, with four edges and five nodes.

Each node of (the complex of bounded cells of) a tropical plane L is labeled by a connected rank-3-matroid. This is the matroid whose matroid polytope is dual to that node in the matroid subdivision of $\Delta(3,n)$ given by L. For $n = 6$ only three classes of matroids occur as node labels of generic planes. These matroids are denoted $\{A, B, C, D\}$, $[A, B, C, D](E)$, or $\langle A; a; (b, c, d, e)\rangle$. Here capital letters are non-empty subsets of and lower-case letters are elements of the set $\{1, 2, 3, 4, 5, 6\}$. All three matroids are graphical. The corresponding graphs are shown in Figure 7.7. Note that an edge labeled with a set of l points should be considered as l parallel edges each labeled with one element of the set.

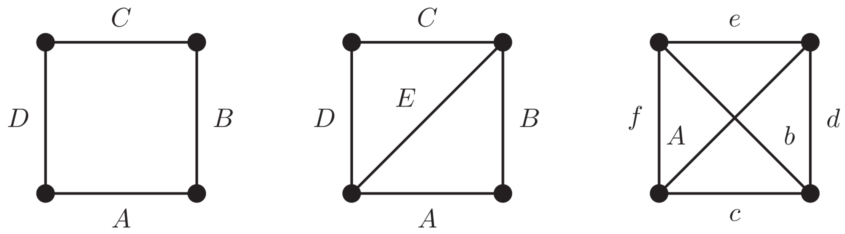

FIGURE 7.7. The graphic matroids corresponding to the labels $\{A, B, C, D\}$, $[A, B; C, D](E)$ and $\langle A; b; (c, d, e, f)\rangle$ used for the nodes in Figure 7.1.

The underlying graph of the matroid $\langle A; b; (c, d, e, f)\rangle$ is the complete graph K_4. The set A is a singleton, and thus its automorphism group is the full symmetric group Sym_4 of order 24 acting on the four nodes of K_4. This matroid occurs in the unique orbit of planes (of type EEEE) in $\mathbb{TP}^6$ whose bounded parts are not two-dimensional. The series-parallel planes use only the matroids $\{A, B, C, D\}$ and $[A, B; C, D](E)$ for their labels.

The **third answer** to our question is the synthesis of the previous two: **draw both** the bounded complex and the tree arrangement. The two pictures can be connected, by linking each node of L to the adjacent unbounded rays and 2-cells. This leads to an accurate diagram of the tropical plane L. The reader might enjoy drawing these connections between the seven rows of Table 2 and the seven pictures in Figure 7.1.

The analogous complete description for $n = 7$ is a main contribution of this paper. Based on the computational results in Section 2, we prepared an online census of $\mathrm{Gr}(3, 7)$ and $\mathrm{Dr}(3, 7)$, with a picture for each bounded complex. This is posted at our website

`www.uni-math.gwdg.de/jensen/Research/G3_7/grassmann3_7.html`.

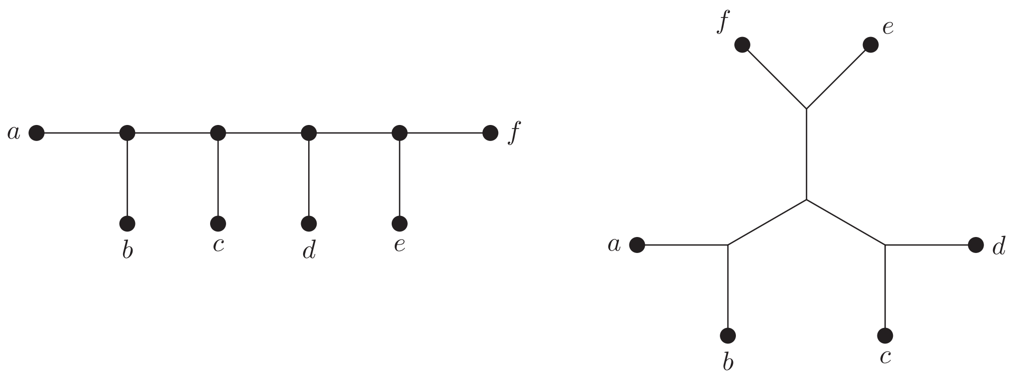

FIGURE 7.8. Caterpillar tree $\mathrm{C}(ab, cd, ef)$ and snowflake tree $\mathrm{S}(ab, cd, ef)$.

The maximal cells of the Dressian $\mathrm{Dr}(3,7)$ correspond to arrangements of seven trivalent trees. As part of our computations, we found that for $n = 7$ there is no difference between abstract tree arrangements and metric tree arrangements: nothing like Example 7.18 exists in this case. To draw the tree arrangements, we note that there are two distinct trivalent trees on six leaves. These are the *caterpillar* and the *snowflake* trees depicted in Figure 7.8. Caterpillar trees exist for all $n \geq 5$, and are encoded using a natural generalization of the notation in Figure 7.6. Note, for instance, the caterpillars with eight leaves in Table 1.

We conclude with a brief discussion of the 94 generic planes depicted on our website. Four types of node labels occur in the Dressian $\mathrm{Dr}(3,7)$. First of all, the matroids $\{A, B, C, D\}$, $[A, B, C, D](E)$, and $\langle A; b; (c, d, e, f)\rangle$ appear again. Here capital letters are non-empty subsets of and lower-case letters are elements of $\{1, 2, \ldots, 7\}$. The other matroid which occurs is the Fano matroid $\mathcal{F}_3$ arising from the projective plane $\mathrm{PG}_2(2)$; see Figure 7.2 (left). It corresponds to the six-dimensional cells of $\mathrm{Dr}(3,7)$ generated by seven vertex splits. Each such 6-cell admits seven coarsenings arising from omitting one of the seven splits. These coarsenings correspond to the non-Fano matroid; see Figure 7.2 (right).

7.6. Restricting to Pappus

The Grassmannian $\mathrm{Gr}(d,n)$ is a variety and the Dressian $\mathrm{Dr}(d,n)$ is a prevariety. We now consider these two inside the tropical projective space $\mathbb{TP}^{\binom{n}{d}-1}$. That projective space is a simplex, and it makes sense to study their intersections with each (relatively open) face of $\mathbb{TP}^{\binom{n}{d}-1}$. That intersection is non-empty only if the face corresponds to a matroid $\mathcal{M}$ of rank d on $[n]$. This leads to the following relative versions of our earlier definitions.

We define the *Grassmannian* $\mathrm{Gr}(\mathcal{M})$ *of a matroid* $\mathcal{M}$ to be the tropical variety defined by the ideal $I_{\mathcal{M}}$ which is obtained from the Plücker ideal by setting to zero all variables p_X where X is not a basis of $\mathcal{M}$. We define the *Dressian* $\mathrm{Dr}(\mathcal{M})$ to be the tropical prevariety given by the set of quadrics which are obtained from the quadratic Plücker relations by setting to zero all variables p_X where X is not a basis of $\mathcal{M}$. Equivalently, in the language of [**29, 30**], the Dressian $\mathrm{Dr}(\mathcal{M})$ is the set of all real-valued valuations of the matroid $\mathcal{M}$. As before, $\mathrm{Gr}(\mathcal{M})$ is a subfan of the Gröbner fan of $I_{\mathcal{M}}$, the Dressian $\mathrm{Dr}(\mathcal{M})$ is a subfan of the secondary fan of the matroid polytope of $\mathcal{M}$, and we regard these fans as polyhedral complexes after removing the lineality space and intersecting with a sphere. Note that the cells of $\mathrm{Dr}(\mathcal{M})$ are in bijection with the regular matroid subdivisions of the matroid polytope of $\mathcal{M}$. The Grassmannian $\mathrm{Gr}(d,n)$ and the Dressian $\mathrm{Dr}(d,n)$ discussed in the previous sections are special cases where $\mathcal{M}$ is the uniform matroid of rank d on n elements. The Dressian $\mathrm{Dr}(d,n)$ contains the Dressians of all matroids of rank d on n elements as subcomplexes at infinity.

In this final section we examine these concepts in detail for one important example, namely, we take $\mathcal{M}$ to be the *Pappus matroid*. Here $d = 3$, $n = 9$, $\mathcal{M}$ has 75 bases, and the non-bases are the nine lines in the Pappus configuration shown in Figure 7.9 (left):

$$123,\ 148,\ 159,\ 247,\ 269,\ 357,\ 368,\ 456,\ 789\,.$$

The ideal $I_{\mathcal{M}}$ is the ideal in the polynomial ring in 75 variables obtained from the Plücker ideal by setting the corresponding nine Plücker coordinates to zero: $p_{123} = \cdots = p_{789} = 0$.

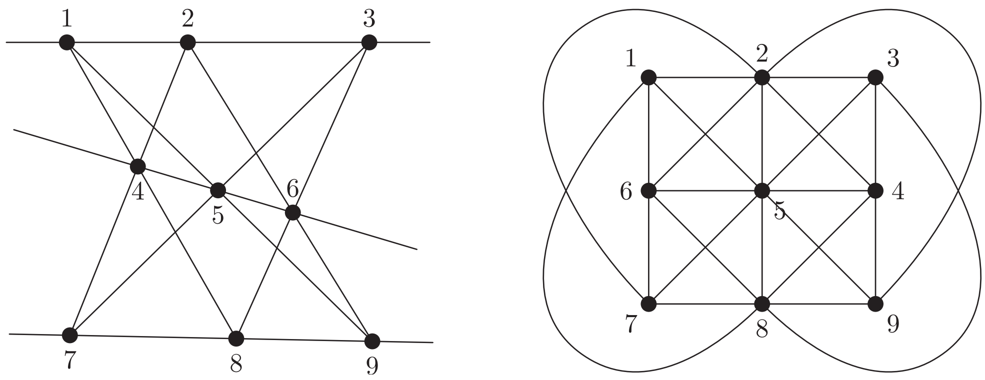

FIGURE 7.9. Pappus configuration (left) and Hessian configuration (right).

The realization space of the Pappus configuration modulo projective transformations is two-dimensional, and the Grassmannian $\mathrm{Gr}(\mathcal{M})$ is the corresponding tropical surface. We shall determine the underlying graph and how it embeds into the Dressian $\mathrm{Dr}(\mathcal{M})$.

PROPOSITION 7.22. *The Grassmannian* $\mathrm{Gr}(\mathcal{M})$ *of the Pappus matroid* $\mathcal{M}$ *is a graph with* 19 *nodes and* 30 *edges. One of the nodes gets replaced by a triangle in the Dressian* $\mathrm{Dr}(\mathcal{M})$. *The Dressian* $\mathrm{Dr}(\mathcal{M})$ *is a simplicial complex with* 18 *vertices,* 30 *edges and one triangle.*

PROOF. What follows is a detailed description, first of $\mathrm{Gr}(\mathcal{M})$ and later of $\mathrm{Dr}(\mathcal{M})$. The Grassmannian $\mathrm{Gr}(\mathcal{M})$ has three *split nodes*, represented by the bases 167, 258 and 349 of the Pappus matroid $\mathcal{M}$. These three bases are characterized by the property that their 2-element subsets form 2-point lines. The corresponding matroid subdivisions are vertex splits, and they are the only splits of the matroid polytope $P_{\mathcal{M}}$. The three split vertices have valence four, and they are connected to a special trivalent *core node* C.

The remaining 15 nodes are all trivalent in $\mathrm{Gr}(\mathcal{M})$, and their subgraph corresponds to the vertices and edges of the complete bipartite graph $K_{3,3}$. The six vertices of $K_{3,3}$ correspond to six *Graves nodes*, one for each of the Graves triads in the Pappus configuration. A *Graves triad* is a partition of the nine points into three bases whose 2-element subsets span 3-point lines. Each Graves node defines a matroid subdivision with three maximal cells. The three corresponding matroids have 52 bases, and they are obtained geometrically by merging together the three points in a triple of the Graves triad. For example, the first matroid in the subdivision defined by the Graves triad $\{145, 237, 689\}$ is obtained from the Pappus matroid by making one, four, and five parallel elements.

The six Graves triads form the vertices of the graph $K_{3,3}$ shown in Figure 7.10. On each of the nine edges lies a *connector node* of $\mathrm{Gr}(\mathcal{M})$, which is between two Graves nodes and also adjacent to one of the three split nodes. Each connector node defines a matroid subdivision with seven maximal cells. The number of bases of these seven matroids are 36, 36, 36, 40, 40, 40, 51. For a concrete example consider the two adjacent Graves triads $\{145, 237, 689\}$ and $\{189, 236, 457\}$. On the edge between them in $K_{3,3}$ we find a connector node which is also adjacent to the split node 167. The seven matroids

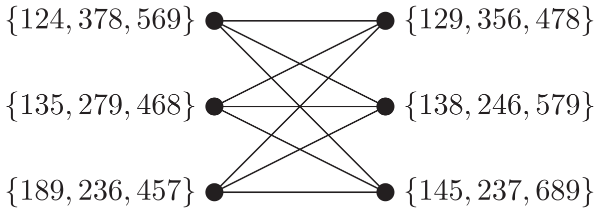

FIGURE 7.10. Complete bipartite graph formed from the Graves triads.

in the matroid subdivision of $P_{\mathcal{M}}$ represented by that connector node are the rows in the following table:

number of bases	parallelism classes
51	{8, 9}, {2, 3}, {4, 5}
40	{2, 3, 6, 7}
40	{1, 4, 5, 7}
40	{1, 6, 8, 9}
36	{4, 5, 7}, {6, 8, 9}
36	{1, 4, 5}, {2, 3, 6}
36	{1, 8, 9}, {2, 3, 7}

We now come to the Dressian $\mathrm{Dr}(\mathcal{M})$ of the Pappus matroid $\mathcal{M}$. This is a non-pure complex whose facets are one triangle and 27 edges. It is obtained from $\mathrm{Gr}(\mathcal{M})$ by removing the core node and replacing it with the *core triangle* whose nodes are the split nodes 167, 258 and 349. Thus $\mathrm{Dr}(\mathcal{M})$ has 18 vertices, 30 edges and one triangle. The core triangle of $\mathrm{Dr}(\mathcal{M})$ represents the matroid subdivision which is obtained from the Pappus matroid polytope by slicing off the three vertices 167, 258 and 349. What remains is the matroid polytope of the Hessian configuration shown in Figure 7.9. This is the matroid associated with the affine plane over the field $\mathbb{GF}(3)$ with three elements. Collinearity of any eleven of its twelve triples implies collinearity of the last. It is this incidence theorem which explains the difference between $\mathrm{Gr}(\mathcal{M})$ and $\mathrm{Dr}(\mathcal{M})$. An algebraic witness is offered by the expression

$$p_{289}p_{389}p_{489}p_{569}p_{589}\underline{p_{167}} - p_{189}p_{389}p_{489}p_{569}p_{679}\underline{p_{258}} + p_{189}p_{289}p_{569}p_{589}p_{678}\underline{p_{349}}\,.$$

This trinomial lies in the Pappus ideal $I_{\mathcal{M}}$, and it shows that the tropical variety of $I_{\mathcal{M}}$ does not contain the entire triangular cone spanned by the basis vectors $e_{167}, e_{258}, e_{349}$. As the minimum must be attained at least twice, we conclude that, locally on the core triangle of the Dressian $\mathrm{Dr}(\mathcal{M})$, the Grassmannian $\mathrm{Gr}(\mathcal{M})$ looks like a tropical line. □

CHAPTER 8

Loose Ends

8.1. Ehrhart Theory and Commutative Algebra

In this section, we will explain the relation of Ehrhart theory and commutative algebra to polytope triangulations. The focus is laid on the relation of the face numbers of faces of the triangulation and similar data in the other fields. The h-vector $h = (h_0, \ldots, h_d)$ of a triangulation Σ of a d-dimensional polytope P is formally defined as $h_k := \sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k} f_{i-1}$, where f_i is the number of i-dimensional faces of Σ (where the empty face is considered to have dimension -1). Since $f_k = \sum_{i=0}^{k+1}\binom{d+1-i}{k+1-i} h_i$ the f- and the h-vector depends monotonously on each other.

We will first describe basic notions and results in Ehrhart theory and commutative algebra and the relation of this topics to regular triangulations of polytopes. In the last subsection, we attempt to generalize some of this to arbitrary regular subdivisions. In particular, we will give an algebraic criterion for the coherency of weight functions.

8.1.1. Ehrhart Theory. Let $P \subset \mathbb{R}^d$ be a polytope with rational vertices. For a positive integer n we define

$$i(P, l) \;=\; |\{x \in P \mid lx \in \mathbb{Z}^l\}| \, .$$

These numbers were first studied by Ehrhart [**34**]. The formal power series

$$\mathrm{Ehr}_P(t) \;:=\; \sum_{i=1}^{\infty} i(P, i) t^i$$

is called the *Ehrhart series* of P.

A polytope with integral vertices is called a *lattice polytope*. It is a classical result of Ehrhart (conjectured in [**32**, Théorème 5 (conjecture)] and proved in [**33**, Théorème 1]) that for all lattice polytopes P there exists a polynomial L_P such that $L_P(l) = i(P, l)$ for all $l \in \mathbb{N}_0$. This polynomial is called the *Ehrhart polynomial* of P. From this one can deduce (see e.g., [**87**, Theorem 2.1]) that there exists a vector $h \in \mathbb{N}_0^{d+1}$ such that

$$\mathrm{Ehr}_P(t) \;=\; \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^{d+1}} \, .$$

We call this vector h the *Ehrhart h-vector* of P. It can be computed from the Ehrhart series as follows.

Lemma 8.1. *Let P be a lattice polytope with Ehrhart series* $\mathrm{Ehr}_P(t) := \sum_{i=1}^{\infty} i(P,i)t^i$. *Then the Ehrhart h-vector of P is given by*

$$h_i = \sum_{k=0}^{i} (-1)^k \binom{d+1}{k} i(P, i-k) .$$

Proof. By definition, we have

$$\begin{aligned} \sum_{i=0}^{d} h_i t^i &= (1-t)^{d+1} \cdot \mathrm{Ehr}_P(t) \\ &= \left(\sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} t^i \right) \left(\sum_{i=1}^{\infty} i(P,i) t^i \right) \\ &= \sum_{i=1}^{\infty} \left(\sum_{k=0}^{i} \binom{d+1}{k} i(P, i-k) \right) t^i . \end{aligned}$$

This shows the claim. □

The following consequence of a theorem by Stanley [**87**, Corollary 2.5] gives us now the relation to polytope triangulations. A triangulation Σ of a lattice polytope is called *unimodular*, if all $S \in \Sigma$ have normalized volume one.

Theorem 8.2. *Let $P \subset \mathbb{R}^d$ be a lattice polytope and Σ a unimodular triangulation of P. Then the h-vector of Σ equals the Ehrhart h-vector of P.*

Since the Ehrhart h-vector of a polytope P is independent of the choice of a triangulation Σ, this implies the following.

Corollary 8.3. [**87**, Corollary 2.7] *All unimodular triangulations of a lattice polytope have the same f-vector.*

The results of Stanley can suitably be generalized for non-unimodular triangulations (see e.g., [**74**, Theorem 1.1]) and this leads to the following.

Corollary 8.4. *Let P be a polytope and Σ a triangulation of P. Then the Ehrhart h-vector of P is an upper bound for the h-vector of Σ.*

The following lemma says that maximizing the Ehrhart h-vector is equivalent to maximizing the Ehrhart series:

Lemma 8.5. *For a polytope P, the coefficients of the Ehrhart series of P depend monotonously on the Ehrhart h-vector of P.*

Proof. By direct computation: One has

$$\sum_{i=1}^{\infty} i(P,i)t^i = \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^{d+1}} = \sum_{i=0}^{d} h_i t^i \left(\sum_{j=0}^{\infty} t^j \right)^{d+1} ,$$

and since all coefficients on the right are positive, one get that increasing some h_i increases also the $i(P,i)$. □

Drawn together with Corollary 8.4 and the fact that the f-vector of a triangulation Σ of a polytope P depends monotonously on the h-vector, Lemma 8.5 implies that the maximal possible f-vector of Σ can read of from the Ehrhart series of P.

8.1.2. Commutative Algebra. We first recall the definitions and results from commutative algebra we will use in the sequel. For details, we refer to the textbooks of Cox, Little, and O'Shea [**18**] and Sturmfels [**89**].

Let $\mathbb{K}$ be an arbitrary field and X a set with n elements. By $\mathbb{K}[X]$ we denote the polynomial ring over $\mathbb{K}$ with the n indeterminates $x \in X$. Sometimes we identify the elements of X with $x_1, \ldots, x_n$. The monomial $x^a := \prod_{i=1}^n x_i^{a_i}$ can be identified with the vector $a = (a_1, \ldots a_n)$ in $\mathbb{N}^n$. A partial order $\prec$ on $\mathbb{N}^n$ is called a *partial term order* if $a \prec b$ implies $a+c \prec b+c$ for all $a, b, c \in \mathbb{N}^n$ and $\prec$ is a well-ordering (i.e., each non-empty subset of $\mathbb{N}$ has a unique smallest element). A partial term order which is a total order is called a *term order*. A weight vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{\geq 0}$ defines a partial term order $\prec_w$ by letting $a \prec_w b$ if and only if $\langle a, w\rangle < \langle b, w\rangle$. If one restricts oneself to homogeneous elements of $\mathbb{K}[X]$, what we will do in the sequel, this also true for arbitrary $w \in \mathbb{R}^n$. For an ideal $I \subset \mathbb{K}[X]$ the *radical* $\mathrm{rad}(I)$ of I is defined as $\mathrm{rad}(I) := \{f \mid f^n \in I \text{ for some } n \in \mathbb{N}\}$.

For a polynomial $f \in \mathbb{K}[X]$ and a partial term order $\prec$ we define its *initial term* $\mathrm{init}_\prec f$ to be the sum of all monomial of f which are minimal with respect to $\prec$. If $\prec$ is a term order, then $\mathrm{init}_\prec f$ is a monomial, the *initial monomial* of f. For an ideal $I \subset k[x]$ the *initial ideal* with respect to $\prec$ is defined as the ideal generated by all initial terms.

$$\mathrm{init}_\prec I = \langle \mathrm{init}_\prec f \mid f \in I\rangle \ .$$

If $\prec$ is a term order, this initial ideal is a *monomial ideal*, that is, an ideal consisting only of monomials. For a weight vector w we also write init_w instead of $\mathrm{init}_{\prec_w}$.

A finite set $\mathcal{G} \subset I$ is called a *Gröbner basis* for an ideal $I \subset \mathbb{K}[X]$ with respect to a term order $\prec$ if it generates I and $\{\mathrm{init}_\prec g \mid g \in \mathcal{G}\}$ generates $\mathrm{init}_\prec I$. A *universal Gröbner basis* $\mathcal{U}$ is a set of polynomials which is a Gröbner basis with respect to all possible term orders. Since an ideal $I \subset \mathbb{K}[X]$ has only finitely many distinct initial ideals [**89**, Theorem 1.2] every ideal has a universal Gröbner basis.

In fact, there is a much richer structure on the set of initial ideals of an ideal I. For a weight vector $w \in \mathbb{R}^n$ one defines the *Gröbner cone* of w as

$$C[w] \ := \ \{w' \in \mathbb{R}^n \mid \mathrm{init}_w(I) = \mathrm{init}_{w'}(I)\} \ .$$

It can be shown that $C[w]$ is indeed a polyhedral cone [**89**, Proposition 2.4] and that the set of all $C[w]$ (for all w) forms a polyhedral fan [**89**, Proposition 2.4], the *Gröbner fan* of the ideal $I \subset \mathbb{K}[X]$. Even more is true.

THEOREM 8.6. [**89**, Theorem 2.5] *If $I \subset \mathbb{K}[X]$ is a homogeneous ideal, then there exists a polytope P called the* state polytope *of I whose normal fan coincides with the Gröbner fan.*

For a monomial ideal $I \subset \mathbb{K}[X]$ the *Hilbert series* of I is defined as

$$\mathrm{Hilb}_I(t) \ = \ \sum_{m \notin I} t^{\deg(m)} \ ,$$

where the sum runs over all monomials not in I and $\deg(m)$ denotes the total degree of the monomial m. For a non-monomial ideal I the *Hilbert series* is defined as the Hilbert series of a monomial initial ideal of I.

The Hilbert series, like the Ehrhart series, can be written in a nice form:

PROPOSITION 8.7. [**66**, Theorem 5.2.20] *The Hilbert Series of an ideal $I \subset \mathbb{K}[X]$ can be written in the form*

$$\mathrm{Hilb}_I(t) \;=\; \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^{d+1}}$$

for some maximal $d \in \mathbb{N}$ (called the dimension of the ideal).

The vector $h = (h_0, h_1, \ldots, h_d)$ in the previous proposition is called the *h-vector* of the ideal I. It follows also that there exists a polynomial H_I such that

$$\mathrm{Hilb}_I(t) \;=\; \sum_{i=0}^{\infty} H_I(i) t^i.$$

This polynomial is called the *Hilbert polynomial* of I. The function $i \mapsto \mathrm{Hilb}_I(i)$ is sometimes called the *Hilbert function* of I. In the same way as in Lemma 8.5 we get.

LEMMA 8.8. *The coefficients of the Hilbert series of an ideal depend monotonously on the h-vector of that ideal.*

LEMMA 8.9. *Let $I \subset \mathbb{K}[X]$ be an ideal. Then* $\mathrm{Hilb}_I = \mathrm{Hilb}_{\mathrm{rad}\, I}$ *if and only if* $I = \mathrm{rad}\, I$.

PROOF. This follows directly form the definition: If the radical is strictly greater then the ideal, then there exists some term in $\mathrm{rad}\, I$ which is not in I, hence the Hilbert function increases. □

We will now relate the Gröbner cone and the state polytope with the secondary cone and the secondary polytope for special ideals. Therefore we define two special classes of ideals: one associated to polytopes and one associated to triangulations, or, more general, to simplicial complexes.

DEFINITION 8.10.
(a) Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite subset of $\mathbb{Z}^d$. Consider the polynomial rings $\mathbb{K}[\mathcal{A}]$ and $\mathbb{K}[T]$ with $T = \{t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}\}$. The kernel of the map
$$\pi : \mathbb{K}[\mathcal{A}] \to \mathbb{K}[T] : a_i \mapsto T^{a_i} \;=\; t_1^{a_{i1}} \cdots t_d^{a_{id}}$$
is called the *toric ideal of $\mathcal{A}$*.

If P is a lattice polytope, we set the *toric ideal of P* as $I_P := I_{\mathrm{Vert}\, P}$.

(b) Let Σ be a simplicial complex with vertex set V. Then the *Stanley-Reisner ideal* I_Σ of Σ is the ideal in $\mathbb{K}[V]$ generated by all monomials $\prod_{v \in F} v$ where $F \subset V$ is not a face of Σ.

On the other hand, given a squarefree monomial ideal $I \subset \mathbb{K}[X]$ the *Stanley-Reisner simplicial complex* Σ_I of I is the simplicial complex on X whose minimal non-faces are the generators of I.

The existence of the Stanley-Reisner simplicial complex and that the two constructions are inverse to each other follows from [**71**, Theorem 1.7]. Furthermore, it is shown in [**89**, Lemma 4.14] that the ideal $I_{\mathcal{A}}$ is homogeneous if and only if $\langle a_i, w \rangle = 1$ for some $w \in \mathbb{Q}^n$ and all $a_i \in \mathcal{A}$. So if $\mathcal{A}$ is a point configuration without multiple points (especially the set of vertices of a polytope P) satisfying our usual assumptions,the corresponding toric ideal is homogeneous.

For toric ideals, the following relation between the Ehrhart series and the Hilbert series was remarked by Sturmfels [**88**, p. 133].

LEMMA 8.11. *Let P be a lattice polytope. Then $H_{I_P}(l) \leq L_P(l)$ for all $l \in \mathbb{N}_0$.*

The correspondence between h-vectors of triangulations and h-vectors of Stanley-Reisner ideals is given in the following proposition.

PROPOSITION 8.12. [**71**, Corollary 1.15] *Let Σ be a simplicial complex and I_Σ its Stanley-Reisner ideal. Then the h-vector of I_Σ equals the h-vector of Σ.*

We are now ready to state the results of Sturmfels [**90**] about the relationship between regular subdivisions of polytopes and initial ideals of toric ideals.

THEOREM 8.13. [**90**, Theorem 3.1] *Let P be a polytope with n vertices and $\omega \in \mathbb{R}^n$ be a weight vector that defines a term order on I_P. Then the polyhedral subdivision $\Sigma_w(P)$ is a regular triangulation of P whose Stanley-Reisner ideal equals the radical of the initial ideal $\mathrm{init}_w I_P$.*

COROLLARY 8.14. [**90**, Corollary 3.2] *The Gröbner fan of I_P is a refinement of the secondary fan of P.*

COROLLARY 8.15. [**90**, Corollary 3.3] *The secondary polytope of P is a Minkowski summand of the state polytope of the toric ideal I_P.*

EXAMPLE 8.16. We consider the square $C_2 \subset \mathbb{R}^3$ whose vertices are the columns of the matrix

$$V^\mathsf{T} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

One easily computes that the corresponding toric ideal $I_{C_2} = \langle x_1x_4 - x_2x_3\rangle$. The possible initial ideals are $\langle x_1x_4\rangle$ and $\langle x_2x_3\rangle$. They correspond to the two possible triangulations of the square. Hence the state polytope of I_P and the secondary polytope of C_2 are line segments.

Note that Theorem 8.13 implies that the Gröbner cone $C[w]$ of a weight vector equals the secondary cone $S[w]$ if and only if the initial ideal $\mathrm{init}_w I_P$ equals $\mathrm{rad}(\mathrm{init}_w I_P)$. A monomial ideal I is called *squarefree*, if all exponents in the generators of I are equal to 0 or 1. It is easily seen, that a monomial ideal I, is squarefree if and only if $I = \mathrm{rad}\, I$ [**66**, Corollary 4.1.12]. Since $\mathrm{init}_w I_P$ is monomial if w is generic, we get the following.

PROPOSITION 8.17. *The Gröbner cone $C[w]$ of a generic weight function w equals the secondary cone $S[w]$ of the corresponding lifting function if and only if the initial ideal $\mathrm{init}_w I_P$ is squarefree.*

COROLLARY 8.18. *Let P be a lattice polytope.*

(a) If there exists some weight vector w such that $\Sigma_w(P)$ is a unimodular triangulation, then the Ehrhart polynomial of P equals the Hilbert polynomial of I_P. Furthermore, the initial ideal $\mathrm{init}_w I_P$ is squarefree.

(b) *Suppose that for some weight vector* w *the initial ideal* $\operatorname{init}_w I_P$ *is monomial and squarefree. Then the Ehrhart polynomial of* P *equals the Hilbert polynomial of* I_P *and* $\Sigma_w(P)$ *is a unimodular triangulation.*

PROOF. (a) Consider the initial ideal $\operatorname{init}_w I_P$ and its Hilbert polynomial $H_{\operatorname{init}_w I_P}$. By definition, $H_{\operatorname{init}_w I_P} = H_{I_P}$. One the other hand, the h-vector of $\operatorname{rad}(\operatorname{init}_w I_P)$ equals the h-vector of $\Sigma_w(P)$ by Theorem 8.13. However, by Theorem 8.2, the h-vector of $\Sigma_w(P)$ equals the Ehrhart h-vector of P. So we have $L_P = H_{\operatorname{rad}(\operatorname{init}_w I_P)}$. Since $\operatorname{rad}(\operatorname{init}_w I_P) \subset \operatorname{init}_w I_P$, the h-vector of $\operatorname{rad}(\operatorname{init}_w I_P)$ is smaller then the h–vector of $\operatorname{init}_w I_P$, and, by Lemma 8.8, we have $L_P = H_{\operatorname{rad}(\operatorname{init}_w I_P)} \leq H_{\operatorname{init}_w I_P} = H_{I_P}$. But, by Lemma 8.11, we have that $H_{I_P} \leq L_P$, so we can conclude $L_P = H_{I_P}$ and $H_{\operatorname{rad}(\operatorname{init}_w I_P)} = H_{\operatorname{init}_w I_P}$.

Furthermore, we get that $\operatorname{rad}(\operatorname{init}_w I_P)$ and $\operatorname{init}_w I_P$ have the same Hilbert function, so they are equal by Proposition 8.9. This shows that $\operatorname{init}_w I_P$ is squarefree.

(b) Since $\operatorname{init}_w I_P$ is squarefree, we have $\operatorname{rad}(\operatorname{init}_w I_P) = \operatorname{init}_w I_P$. Using the same argumentation as above, we directly get $L_P = H_{\operatorname{rad}(\operatorname{init}_w I_P)} = H_{\operatorname{init}_w I_P} = H_{I_P}$. Furthermore, by Theorem 8.13 in connection with Proposition 8.12, the h-vector of $\Sigma_w(P)$ equals the h-vector of P. Since only unimodular triangulations can have this maximal possible h-vector, $\Sigma_w(P)$ is a unimodular triangulation. □

A lattice polytope P is called *unimodular*, if all its triangulations are unimodular. In this special case we can conclude from Proposition 8.17 and Corollary 8.18.

COROLLARY 8.19. *Let* P *be a unimodular polytope. Then we have:*

(a) *For any generic weight vector* w *the initial ideal* $\operatorname{init}_w I_P$ *is a squarefree monomial ideal.*

(b) *The Gröbner fan of* I_P *equals the secondary fan of* P *and the state polytope of* I_P *equals the secondary polytope of* P.

EXAMPLE 8.20. An example of a unimodular polytope is the product of two simplices. This application was discussed in detail in [**88**, Section 6].

We now return to the case of a general lattice polytope P.

PROPOSITION 8.21. *Let* P *be a lattice polytope and* w *any generic weight function for* P. *An upper bound for the* h*-vector of triangulations* $\Sigma_w(P)$ *is given by the* h*-vector of* I_P. *The corresponding* f*-vector gives a bound for the* f*-vector of* $\Sigma_w(P)$. *This bound is achieved if and only if* $\Sigma_w(P)$ *is unimodular.*

PROOF. As in the proof of Corollary 8.18, we get that the h-vector of $\operatorname{rad}(\operatorname{init}_w I_P)$ is (componentwise) smaller or equal then the h-vector of I_P. But the h-vector of $\operatorname{rad}(\operatorname{init}_w I_P)$ equals the h-vector of $\Sigma_w(P)$ by Theorem 8.13 together with Proposition 8.12. So the h-vector of I_P is an upper bound and the same is true for the f-vector. The last statement follows from Corollary 8.18. □

8.1.3. Relations to Coherency of Weight Functions. Our aim is now to find out how the notion of coherency of weight functions introduced in Section 2.2 fits into the relation to commutative algebra.

For a polytope P and a weight function $w : \operatorname{Vert} P \to \mathbb{R}$, we consider the initial ideal $I_w(P) := \operatorname{init}_w I_P$. If $I_w(P)$ is a monomial ideal, we have an associated simplicial complex, the Stanley-Reisner simplicial complex $\Sigma_{\operatorname{rad} I_w}$, which is isomorphic to the triangulation $\Sigma_w(P)$ of P. We aim at a similar relation for weight functions w that do not necessarily define a term order. Therefore, we define the corresponding subdivision $\Sigma_{I_w}(P)$ of P as the common coarsening of all subdivision corresponding to monomial initial ideals of $I_w(P)$.

Lemma 8.22. *For a weight vector w on a polytope P we have $\Sigma_{I_w}(P) = \Sigma_w(P)$.*

Proof. Consider the set $\mathcal{I}$ of all monomial initial ideals of $I_w(P)$. For each $I \in \mathcal{I}$ there exists a maximal Gröbner cone C_I corresponding to I. By Theorem 8.14, each such cone C_I is contained in a unique secondary cone S_I. Our definition of $\Sigma_{I_w}(P)$ just states that $\Sigma_{I_w}(P) = \Sigma_{w'}(P)$, where w' is some weight function for P with $S[w'] = \bigcap_{I\in\mathcal{I}} S_I$. However, we have $w \in \bigcap_{I\in\mathcal{I}} C_I \subset \bigcap_{I\in\mathcal{I}} S_I$ and hence $S[w] = S[w']$. This shows the claim. □

We need the following two statements about radical and initial ideals.

Lemma 8.23. *Let I be a homogeneous ideal in a polynomial ring $\mathbb{K}[X]$ and $w : X \to \mathbb{R}$ a weight vector. Then we have*

$$\operatorname{rad}(\operatorname{init}_w(\operatorname{rad} I)) = \operatorname{rad}(\operatorname{init}_w I).$$

Proof. Since $\operatorname{rad} I \supseteq I$ we obviously have $\operatorname{rad}(\operatorname{init}_w(\operatorname{rad} I)) \supseteq \operatorname{rad}(\operatorname{init}_w I)$, so it remains to prove that $\operatorname{rad}(\operatorname{init}_w(\operatorname{rad} I)) \subseteq \operatorname{rad}(\operatorname{init}_w I)$. Let $f \in \operatorname{rad}(\operatorname{init}_w(\operatorname{rad} I))$. This means that there exists some $k_1, k_2 \in \mathbb{N}$ and $g \in \mathbb{K}[X]$ such that $f^{k_1} = \operatorname{init}_w g$ and $g^{k_2} \in I$. However, it is easily seen that for any polynomial $h \in K[X]$ we have $\operatorname{init}_w(h^k) = (\operatorname{init}_w h)^k$. So we have $\operatorname{init}_w(g^{k_2}) = f^{k_1k_2}$, and hence we get that $f^{k_1k_2} \in \operatorname{init}_w I$ which implies $f \in \operatorname{rad}(\operatorname{init}_w I)$. □

Lemma 8.24. *Let I be a homogeneous ideal in a polynomial $\mathbb{K}[X]$ and w, w' weight vectors. Then the following assertions are equivalent.*

(a) $\operatorname{init}_w(\operatorname{init}_{w'} I) = \operatorname{init}_{w'}(\operatorname{init}_w I)$,
(b) $\operatorname{init}_w(\operatorname{init}_{w'} I) = \operatorname{init}_{w+w'}(I) = \operatorname{init}_{w'}(\operatorname{init}_w I)$,
(c) the set of monomial initial ideals of $\operatorname{init}_{w+w'}(I)$ *is the intersection of those of* $\operatorname{init}_w(I)$ *and* $\operatorname{init}_{w'}(I)$,
(d) $C[w] \cup C[w'] \subset C[w + w']$.

Proof. Let $\mathcal{U}$ be a universal Gröbner basis of I. Then for an arbitrary weight vector $\tilde{w} : X \to \mathbb{R}$ we have that $\operatorname{init}_{\tilde{w}} I = \langle \operatorname{init}_{\tilde{w}} u \mid u \in \mathcal{U}\rangle$. If (a) holds we have $\tilde{u} := \operatorname{init}_w(\operatorname{init}_{w'} u) = \operatorname{init}_{w'}(\operatorname{init}_w u)$ for all $u \in \mathcal{U}$. However, by the definition of the partial term order associated to a weight vector, we have that $\tilde{u} = \operatorname{init}_{\epsilon_1 w + w'} u = \operatorname{init}_{w+\epsilon_2 w'} u$ for some $\epsilon_1, \epsilon_2 > 0$. This implies $\operatorname{init}_{w+w'} u = \tilde{u}$, so we have shown the equivalence of (a) and (b). Furthermore, (d) is just a reformulation of c in term of the Gröbner fan. Hence it suffices to show (b) $\implies$ (c) and (d) $\implies$ (a).

To prove (b) $\implies$ (c), denote the set of monomial initial ideals of $\operatorname{init}_w I$ by $\mathcal{P}(w)$ and that of $\operatorname{init}_{w'}(\operatorname{init}_w I)$ by $\mathcal{P}(w, w')$. With this one has $\mathcal{P}(w, w') \subseteq \mathcal{P}(w)$, $\mathcal{P}(w', w) \subseteq \mathcal{P}(w')$, and $\mathcal{P}(w) \cap \mathcal{P}(w') \subseteq \mathcal{P}(w, w')$. Now suppose that (b) holds, which means that $\mathcal{P}(w, w') = \mathcal{P}(w + w') = \mathcal{P}(w', w)$. Then $\mathcal{P}(w) \cap \mathcal{P}(w') \subseteq \mathcal{P}(w, w') = \mathcal{P}(w + w') \subseteq \mathcal{P}(w) \cap \mathcal{P}(w')$. This implies (c).

It remains to show (d) $\implies$ (a). That $C[w] \cup C[w'] \subset C[w + w']$ implies that the cones $C[w]$ and $C[w']$ are both faces of $C := C[w + w']$ and $C[w + w']$ is minimal with this property. It is now easily seen, that C has to be the cone corresponding to the initial ideal $\mathrm{init}_w(\mathrm{init}_{w'} I)$ and also to the initial ideal $\mathrm{init}_{w'}(\mathrm{init}_w I)$, hence the two ideals agree. □

We now can give a generalization of Proposition 8.21 to non-generic weight functions.

PROPOSITION 8.25. *Let P be a lattice polytope. The h-vector of $\mathrm{rad}\, I_w$ is an upper bound for the h-vector of all triangulations of P that refine $\Sigma_w(P)$.*

PROOF. This follows from Lemma 8.23 using the same arguments as in the proof of Proposition 8.21. □

Proposition 8.25 may be used to construct triangulations of P with small f-vector: Look at some (or all) coarsest subdivision of P and choose that with the smallest h-vector, then look at different refinements of it, look again at the h-vector, and so on.

As the last result of this section, we give an algebraic criterion for coherency of weight functions.

PROPOSITION 8.26. *Let P be a lattice polytope and w and w' weight functions for P. If $\mathrm{init}_w(\mathrm{init}_{w'} I_P) = \mathrm{init}_{w'}(\mathrm{init}_w I_P)$ then the sum $w + w'$ is coherent.*

PROOF. By definition, the sum $w + w'$ is coherent if the cones $S[w]$ and $S[w']$ in the secondary fan are contained in $S[w + w']$. By Lemma 8.24, the cones $C[w]$ and $C[w']$ of the Gröbner fan are contained in $C[w + w']$. Since the Gröbner fan is a refinement of the secondary fan by Corollary 8.14, this shows the claim. □

8.2. Finite Metric Spaces

In Section 2.6, we discussed the relation of general splits and tight spans to the classical case of finite metric spaces. In this section, we will give some additional results about splits and tight spans of finite metric spaces, motivating a further study of them with the methods developed for general tight spans and splits.

PROPOSITION 8.27. *Let $\mathcal{S}$ be a weakly compatible split system for $\Delta(2, n)$. Then $\mathcal{S}$ is incompatible if and only if it is totally incompatible.*

PROOF. To a split $S = (A, B)$ of $\Delta(2, n)$ we associate the complete bipartite graph G_S on $[n]$ connecting all elements of A with all elements of B. (So i and j are connected in G_S if and only if $e_i + e_j$ lies in the splitting hyperplane H_S.) In this notion, a split system $\mathcal{S}$ is totally incompatible, if and only if the intersection $G_{\mathcal{S}} := \cap_{S \in \mathcal{S}} G_S$ (i.e., the graph on $[n]$ whose edges are the common edges of all G_S, $S \in \mathcal{S}$) of the corresponding bipartite graphs is spanning.

To see this, let first $S_1 = (A_1, B_1)$ and $S_2 = (A_2, B_2)$ be compatible, so we can assume that $A_1 \subsetneq A_2$. Then any $i \in A_2 \setminus A_1$ is an isolated point of $G_{S_1} \cap G_{S_2}$ since in G_{S_1}, the vertex i is only connected with vertices of A_1, in G_{S_2}, it is only connected with elements of B_2, and $A_1 \cap B_2 = \emptyset$. So $G_{S_1} \cap G_{S_2}$ is not spanning.

Let now S_1 and S_2 be incompatible and $i \in A_1 \cap A_2$. Since $B_1 \cap B_2 \neq \emptyset$, there exists some $j \in B_1 \cap B_2$ such that i is connected to j in $G_{S_1} \cap G_{S_2}$, and hence $G_{S_1} \cap G_{S_2}$ is spanning. If i is in any other intersection, we similarly get some j to which i is connected in $G_{S_1} \cap G_{S_2}$. The claim about arbitrary split systems follows by induction.

We will now prove the statement with induction on the cardinality of $\mathcal{S}$. For $|\mathcal{S}| = 2$ the assertion is obviously true for arbitrary polytopes by definition. So assume, that there is an incompatible, but non-totally incompatible split system $\mathcal{S}$ with $|\mathcal{S}| \geq 3$. This means that the intersection $G_{S_1} \cap G_{S_2}$ for any $S_1, S_2 \in \mathcal{S}$ is spanning, but the graph $G_{\mathcal{S}}$ is not. Let $m_1 \in [n]$ be an isolated vertex of $G_{\mathcal{S}}$ and $\mathcal{S}' := \mathcal{S} \setminus \{S_1\}$ for some $S_1 \in \mathcal{S}$. By the induction hypotheses, we have an edge in $G_{\mathcal{S}'}$ connecting m_0 with some $m_1 \in [n]$. By assumption, m_0 and m_1 are not connected by an edge of G_{S_1} so in the notion introduced before Proposition 2.47 we have $S_1(m_0) = S_1(x_1)$, but $S(m_0) \neq S(m_1)$ for all $S \in \mathcal{S}'$. With the same argumentation we find $S_2, S_3 \in \mathcal{S}$ distinct from S_1 and $m_2, m_3 \in [n]$ such that $S_i(m_0) = S_j(m_j)$ if and only if $i = j$ for all $i, j \in [n]$. Hence $\mathcal{S}$ is not weakly compatible by Proposition 2.47. □

An interesting property that can easily be derived with our general theory states that all split triangulations of $\Delta(2, n)$ have the same f-vector (cf. Corollary 8.3).

Proposition 8.28. *All split triangulations of* $\Delta(2, n)$ *are unimodular.*

Proof. Let Σ be a split triangulation of $\Delta(2, n)$. Since the claim is obvious for $n = 3$, we can assume (using induction) that simplices $F \in \Sigma$ that are contained in the boundary of $\Delta(2, n)$ have normalized volume 1. By Proposition 2.39, all splits of $\Delta(2, n)$ are induced by hyperplanes H defined by

$$\sum_{i \in A} x_i = \sum_{i \notin A} x_i$$

for some $A \subsetneq [n]$. So $H \cap \Delta(2, n)$ is the convex hull of all points of the form $e_i + e_j$ with $i \in A, j \in [n] \setminus A$. This is a product of a $(|A| - 1)$- and an $(n - |A| - 1)$-dimensional simplex. It is a well know fact that all triangulations of products of two simplices are unimodular (see e.g., [**6**]). Since any $F \in \Sigma$ that has codimension one is either in the boundary of $\Delta(2, n)$ or in some split hyperplane, F has normalized volume one. However, this implies that all $F \in \Sigma$ must have normalized volume one. □

Remark 8.29.
- ▹ The f-vector for the unimodular triangulations of $\Delta(2, n)$ was computed in [**46**].
- ▹ For $n \leq 5$ all triangulations of the second hypersimplex are unimodular. They are all (up to symmetry) given in [**20**], the corresponding tight spans were previously classified in [**23**].
- ▹ For $n \geq 6$ not all triangulations of the second hypersimplex are unimodular. Sturmfels and Yu [**91**] classified all triangulations of $\Delta(2, 6)$ and found (up to symmetry) 327 unimodular and 12 non-unimodular triangulations.
- ▹ In fact, there is – up to symmetry – only one split triangulation of $\Delta(2, n)$; see Example 2.50. Since this triangulation is unimodular, this gives an alternative proof of Proposition 8.28.

Split systems of $\Delta(2, n)$ have another interesting property:

PROPOSITION 8.30. *The union of two compatible split systems of* $\Delta(2, n)$ *is weakly compatible.*

PROOF. Directly from Proposition 2.47, one sees that each two splits of the second hypersimplex are weakly compatible. (Note that this is obviously not true for arbitrary polytopes, for example for polygons with more than three vertices.) Now the claim follows by Corollary 3.9. However, we additionally give a direct combinatorial proof.

Let the union of two compatible split systems $\mathcal{S}_1$ and $\mathcal{S}_2$ be not weakly compatible. This means that we have three splits $S_1, S_2, S_3 \in \mathcal{S}_1 \cup \mathcal{S}_2$ and four elements m_0, m_1, m_2, m_3 such that $S_i(m_j) = S_i(m_0)$ if and only if $i = j$. We can assume that S_1 and S_2 are in $\mathcal{S}_1$, and so they are compatible by assumption. This means that one of the four intersections $S_1(m_0) \cap S_2(m_0)$, $S_1(m_0) \cap S_2(\bar{m}_0)$, $S_1(\bar{m}_0) \cap S_2(m_0)$ and $S_1(\bar{m}_0) \cap S_2(\bar{m}_0)$ is empty, where $\bar{\cdot}$ denotes the complement of a set in $[n]$. However, we have $m_0 \in S_1(m_0) \cap S_2(m_0)$, $m_1 \in S_1(m_1) \cap S_2(m_1) = S_1(m_0) \cap S_2(\bar{m}_0)$, $m_2 \in S_1(m_2) \cap S_2(m_2) = S_1(\bar{m}_0) \cap S_2(m_0)$ and $m_3 \in S_1(m_3) \cap S_2(m_3) = S_1(\bar{m}_0) \cap S_2(\bar{m}_0)$ by the definition of weak compatibility. This is a contradiction to our assumption and hence finishes the proof. □

If we interpret Proposition 8.30 in the theory of finite metric spaces we get an interesting statement. We start out with some definitions

DEFINITION 8.31. (a) A *weightend tree* $T = (V, E, w)$ is a tree (V, E) with vertices V and edges E together with a weight function $w : E \to \mathbb{R}^+$ assigning a weight to each edge.

(b) Let $T = (V, E, w)$ be a weightend tree. We define an associated metric d_T on the set V by

$$d_T(x, y) = \sum_{i=1}^{k} w(z_i)\,,$$

where $x = z_1, \ldots, z_k = y$ is the (unique) path in T from x to y. A metric d is called a *tree metric* if there exists a weightend tree T such that $d = d_T$.

The key observation about tree metrics and tight spans is the following; see [**23**, Theorem 8] and Proposition 2.30.

LEMMA 8.32. *Let* d_T *be a tree metric.*

(a) The tight span $\mathcal{T}_{-d_T}(\Delta(2, n))$ *of* d_T *is a realization of the tree* $T = (V, E, w)$ *in* $\mathbb{R}^{|V|}$ *whereas the edge length are measured by* $\|\cdot\|_\infty$.

(b) $\mathcal{T}_{-d_T}(\Delta(2, n))$ *is induced by a compatible split system such that each edge of* T *corresponds to one split.*

Now we can give a new proof of the following.

COROLLARY 8.33. [**4**, Corollary 8] *Let* d_{T_1} *and* d_{T_2} *be tree metrics. Then the sum* $d_{T_1} + d_{T_2}$ *is coherent, and* $\mathcal{T}_{-d_{T_1}-d_{T_2}}(\Delta(2, n))$ *is at most two-dimensional.*

PROOF. By Proposition 2.30 the tight span of a metric d is a tree if and only if $\Sigma_{-d}(\Delta(2, n))$ is induced by a compatible split system. So by Proposition 8.30 in connection with Corollary 2.4 the sum $d_{T_1} + d_{T_2}$ is coherent. It follows from Proposition 3.14 that $\mathcal{T}_{-d_{T_1}-d_{T_2}}(\Delta(2, n))$ is at most two-dimensional. □

Since the union of k compatible split systems does not have to be weakly compatible for $k \geq 3$, the argumentation in the proof of Corollary 8.33 cannot be generalized for arbitrary k. So the following question remains.

QUESTION 8.34. Let $\mathcal{B}$ be a set of k metric trees. Is $\mathcal{T}_{-\sum_{T\in\mathcal{B}} d_T}(\Delta(2,n))$ at most k-dimensional?

Corollary 8.33 can be used to give an algorithm to compute the tight span of the sum of two metric trees. We call a set $\mathcal{B}$ of trees compatible if the union of all associated splits system is weakly compatible. In fact, the algorithm we will sketch now computes the tight span of the sum of several compatible tree metrics or an arbitrary weakly compatible split system $\mathcal{S}$ of $\Delta(2,n)$. It turns out that our algorithm is similar to the algorithm given by Dress and Huson in [**28**], so we refer there for the details and only give a very rough sketch.

A weightend tree $T = (V, E, w)$ can be thought of as a collection $\mathcal{S}_T$ of compatible splits S (one for each edge of the tree (Proposition 2.30) occupied with weights w_i (the length of the edge). The tight span $\mathcal{T}_{-\sum_{T\in\mathcal{B}} d_T}(\Delta(2,n))$ of a set of compatible trees then equals $\mathcal{T}(\mathcal{S}) := \mathcal{T}_{\mathcal{S}}(\Delta(2,n))$ where $\mathcal{S} := \bigcup_{T\in\mathcal{B}} \mathcal{S}_T$.

We regard the tight span $\mathcal{T}(\mathcal{S})$ as an abstract polyhedral complex such that each element of $[n]$ is attached to some vertex of $\mathcal{T}(\mathcal{S})$. By [**52**, Corollary 7.3], all cells in the tight span of weakly compatible split system are either isomorphic to cubes or to rhombic dodecahedra. So the full nature of each cell $C \in (\mathcal{S})$ is determined by its vertices and its edges. If we additionally store a weight for each edge, we also have the metric structure of the tight span. The algorithm now works by starting with the abstract polyhedral complex $C_\emptyset$ consisting of an isolated vertex to which all $i \in [n]$ are associated and adding subsequently all $S \in \mathcal{S}$ (in the right order). Given the (abstract) tight span $C_{\mathcal{S}}$ for a weakly compatible split system $\mathcal{S}$ and a split $S = (A, B)$ the complex $C_{\mathcal{S}\cup\{S\}}$ is obtained by first computing the subcomplex S of $C_{\mathcal{S}}$ which separates the elements of A form the elements of B. Then S is duplicated and the two copies of S are connected by edges corresponding to S. If we have a weight w_S associated to each $S \in \mathcal{S}$ (e.g., if $\mathcal{S}$ comes from a set of compatible trees), we mark all these edges with w_S and have at the end also a metric representation of $\mathcal{T}(\mathcal{S})$.

8.3. Quasi Split Subdivisions

In Chapter 6, we studied generalizations of splits by looking at additional facets of the secondary polytope. In particular, in Section 6.2, we defined k-split subdivisions as generalizations of split subdivisions. In this section, we describe another way to define a generalization of split subdivisions. The following examples should serve as a motivation for the definition.

EXAMPLE 8.35. Let P be an octahedron such that one vertex is slightly perturbed. Then the subdivision of P into four simplices defined by the edge connecting to of the other vertices is not a split subdivision of P. However, the same subdivisions of the non-perturbed octahedron would be a split subdivision.

EXAMPLE 8.36. Let Q be the the quadrangle whose vertices are the columns of the matrix

$$V^{\mathsf{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 2 & -1 & -1 \end{pmatrix}.$$

and Σ the triangulation of Q depicted in Figure 8.1. Then the prism over Q has an obvious split subdivision whose cells are the prisms over the cells of Σ. This subdivision cannot be refined to a split triangulation. However, we could use split subdivisions for each cell that fit together to a split triangulation of the prism.

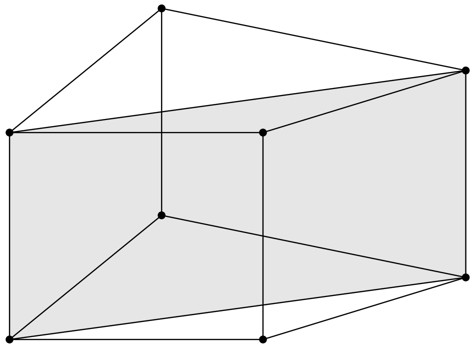

FIGURE 8.1. Product of two split triangulations that cannot be refined to a split subdivision.

We now give a recursive definition of quasi split subdivision.

DEFINITION 8.37. Let P be a polytope and Σ a subdivision of P. Then Σ is called a *quasi split subdivision* if either

- ▹ Σ is a split subdivision or
- ▹ there exists a split S of P and quasi split subdivisions Σ_+, Σ_- of S_+, S_-, respectively, which agree on $P \cap H_S = S_+ \cap S_-$ such that $\Sigma = \Sigma_+ \cup \Sigma_-$.

If we now look again at Example 8.35 and Example 8.36, we see that the subdivisions we constructed there fit with this definition. However, note that if we perturb not only one, but all vertices of the octahedron of Example 8.35 and take a similar triangulation, this would not be a quasi split subdivision. By contrast, Example 8.35 is only a special case of some more general construction: If the factor of a product of polytopes allow split triangulations, the product allows quasi split subdivisions. This will be shown in the rest of this section.

LEMMA 8.38. *Let P, Q be polytopes and Σ_P, Σ_Q split subdivisions of P, Q, respectively. Then*

$$\Sigma_P \times \Sigma_Q := \{F_P \times F_Q \mid F_P \in \Sigma_P, F_Q \in \Sigma_Q\}$$

is a split subdivision of $P \times Q$. If Σ_P, Σ_Q are quasi split subdivisions, then $\Sigma_P \times \Sigma_Q$ is a quasi split subdivision.

PROOF. We will assume throughout the proof that P and Q are full-dimensional and work with affine split hyperplanes.

It is easily seen that $\Sigma_P \times \Sigma_Q$ is a valid subdivision of $P \times Q$ (see e.g., [**19**, Proposition 4.2.14]). To prove that it is a split subdivision if Σ_P and Σ_Q are split subdivisions, it suffices to prove that for any codimension-one-cell $C \in \Sigma_P \times \Sigma_Q$ the hyperplane $H := \operatorname{aff} C$ defines a split of $P \times Q$ and that there exist a set $\mathcal{C}$ of codimension-one-cells of $\Sigma_P \times \Sigma_Q$ such that $H \cap (P \times Q) = \bigcup_{C \in \mathcal{C}} C$.

Since C is of codimension one, it is of the form $C_P \times C_Q$ for some cells $C_P \in \Sigma_P$, $C_Q \in \Sigma_Q$ one of which is full-dimensional and the other has codimension one. We assume that $C_1 := C_P$ is of codimension one. Since Σ_P is a split subdivision, $\operatorname{aff} C_P$ defines a split of P and hence, by Lemma 3.27 (a), H defines a split of $P \times Q$. Now let $C_2, \dots, C_k$ be codimension-one-cells of Σ_P such that $\operatorname{aff} C_1 \cap P = \bigcup_{i=1}^{k} C_i$. These C_i exist since Σ_P is a split subdivision. We define $\mathcal{C} := \{C_i \times F \mid 1 \le i \le k, F \in \Sigma_Q \text{ full-dimensional}\}$. Since $H = \operatorname{aff} C_1 \times \operatorname{aff} Q$, we get $H \cap (P \times Q) = \bigcup_{C \in \mathcal{C}} C$. This shows the claim for split subdivisions.

If Σ_P and Σ_Q are quasi split subdivisions with splits $S(P), S(Q)$ of P, Q, respectively, we have a subdivision of $P \times Q$ into the four maximal dimensional cells $S(P)_+ \times S(Q)_+$, $S(P)_+ \times S(Q)_-$, $S(P)_- \times S(Q)_+$, and $S(P)_- \times S(Q)_-$. The claim now follows from the split subdivision case taking into account the recursive definition. ◻

PROPOSITION 8.39. *Let P, Q be polytopes and Σ_P, Σ_Q quasi split triangulations of P, Q, respectively. Then there exists a quasi split triangulation Σ of $P \times Q$ that is a refinement of $\Sigma_P \times \Sigma_Q$.*

PROOF. As Σ_P, Σ_Q are triangulations, the product $\Sigma_P \times \Sigma_Q$ is a subdivision of $P \times Q$ into products of simplices. By Lemma 8.38, it is a quasi split subdivision. Now take any ordering of the vertices of P and any ordering of the vertices of Q. For each product of simplices $F := \sigma_P \times \sigma_Q \in \Sigma_P \times \Sigma_Q$ with simplices $\sigma_P \in \Sigma_P$, $\sigma_Q \in \Sigma_Q$, we take the staircase triangulation from Example 3.22 that is obtained by the restriction of the orderings of the vertices of P,Q to the vertices of σ_P,σ_Q, respectively. So we have split triangulations for each $F \in \Sigma_P \times \Sigma_Q$ that agree on their intersections by construction. Combining all these triangulations gives the desired quasi split triangulation of $P \times Q$. ◻

Especially, we have the following, which is not true if we omit the "quasi" as we have seen in Example 8.36.

COROLLARY 8.40. *If P and Q are polytopes such that there exists (quasi) split triangulations of P and Q, then there exists a quasi split triangulation of $P \times Q$.*

8.4. Fiber Polytopes

In [**10**], Billera and Sturmfels define the notion of fiber polytopes and show that secondary polytopes are special instances for them. In this section, we will investigate how splits are related to this construction.

For polytopes P, Q and a projection $\pi : Q \to \mathbb{R}^{d+1}$ with $\pi(Q) = P$ the *fiber polytope of Q over P* is defined as

$$\text{FibPoly}(Q, P) := \left\{ \frac{1}{\operatorname{vol} P} \int_P \gamma(x) dx \;\middle|\; \gamma \text{ is a section of } \pi \right\}, \tag{8.1}$$

where a *section* γ of π is a function $\gamma : P \to Q$ with $\pi \circ \gamma = \mathrm{id}_Q$. Instead of this definition as a Minkowski integral of a polytope bundle, we will use the following description, which is given in [**10**, Theorem 1.5]. We denote by $C(\pi)$ the polytopal complex obtained as the common refinement of all subdivisions Σ of P with the property that for all faces $F \in \Sigma$ there exists some face F' of Q with $\pi(F') = F$. One can show that all such subdivisions have to be regular. In general, $C(\pi)$ is not a subdivision of P according to the definition used throughout this thesis since there might occur inner points. However, $C(\pi)$ is a subdivision of some point configuration $\mathcal{A}$ with $\operatorname{conv}\mathcal{A} = P$. Then we have

$$\text{FibPoly}(Q, P) \;=\; \sum_{C \in C(\pi)} \frac{\operatorname{vol} C}{\operatorname{vol} P}\, \pi^{-1}(c_C)\,, \tag{8.2}$$

where the sum runs over all full-dimensional cells C and c_C is the centroid of C.

This description of the fiber polytope gives rise to a natural generalization: Let $\mathcal{A}$ be any point configuration with $\operatorname{conv}\mathcal{A} = P$ and Σ a subdivision of $\mathcal{A}$. Abusing notation, but in accordance with standard terminology, we will call a subdivision of $\mathcal{A}$ with this property also a subdivision of P. There will be no chance for confusion since all subdivisions of P in our usual definition that occur in this section are regular, and we can stick to the term *regular subdivision* for them. We define the *fiber polytope of Q over P with respect to* Σ as

$$\text{FibPoly}(Q, P, \Sigma) \;=\; \sum_{C \in \Sigma} \frac{\operatorname{vol} C}{\operatorname{vol} P}\, \pi^{-1}(c_C)\,. \tag{8.3}$$

LEMMA 8.41. *Let Σ be a subdivision of P and Σ' a refinement of Σ. Then*

$$\text{FibPoly}(Q, P, \Sigma') \;\subseteq\; \text{FibPoly}(Q, P, \Sigma)\,.$$

PROOF. That Σ' is a refinement of Σ means that for each maximal cell C of Σ there exists a set M of maximal cells of Σ' such that $C = \bigcup_{D \in M} D$. So, by our definition (8.3), it suffices to show that

$$\operatorname{vol} P \cdot \pi^{-1}(c_P) \;=\; \sum_{C \in \Sigma} \operatorname{vol} C \cdot \pi^{-1}(c_C)\,,$$

for any subdivision Σ of P. Suppose that $Q \subset \mathbb{R}^{e+1}, P \subset \mathbb{R}^{d+1}$ have codimension one (as usual) and that $\pi : Q \to \mathbb{R}^{d+1}$ is given by $x \mapsto Ax$ for some $(d+1) \times (e+1)$-matrix A. Then for any subpolytope C of P we have

$$\begin{aligned} \operatorname{vol} C \cdot \pi^{-1}(c_C) \;&=\; \left\{\operatorname{vol} C \cdot x \in \mathbb{R}^{e+1} \;\middle|\; Ax = c_C \text{ and } x \in Q\right\} \\ &=\; \left\{x \in \mathbb{R}^{e+1} \;\middle|\; Ax = \operatorname{vol} C \cdot c_C \text{ and } \frac{x}{\operatorname{vol} C} \in Q\right\}. \end{aligned}$$

Since $\sum_{C \in \Sigma} \frac{\operatorname{vol} C}{\operatorname{vol} P} = 1$ we have that $\frac{x_C}{\operatorname{vol} C} \in Q$ for all $C \in \Sigma$ implies

$$\frac{1}{\operatorname{vol} P} \sum_{C \in \Sigma} x_C \;=\; \sum_{C \in \Sigma} \frac{\operatorname{vol} C}{\operatorname{vol} P} \cdot \frac{x_C}{\operatorname{vol} C} \in Q\,.$$

Furthermore, by the definition of the centroid, we have $\sum_{C\in\Sigma} \operatorname{vol} C \cdot c_C = \operatorname{vol} P \cdot c_P$. Hence we can conclude

$$\begin{aligned}\sum_{C\in\Sigma} \operatorname{vol} C \cdot \pi^{-1}(c_C) &= \left\{\sum_{C\in\Sigma} x_C \in \mathbb{R}^{e+1} \,\middle|\, Ax_C = \operatorname{vol} C \cdot c_C \text{ and } \frac{x_C}{\operatorname{vol} C} \in Q \text{ for all } C \in \Sigma\right\} \\ &\subseteq \left\{x \in \mathbb{R}^{e+1} \,\middle|\, Ax = \operatorname{vol} P \cdot c_P \text{ and } \frac{x}{\operatorname{vol} P} \in Q\right\} \\ &= \operatorname{vol} P \cdot \pi^{-1}(c_P)\,.\end{aligned}$$

□

Especially, this implies that $\operatorname{FibPoly}(Q,P) = \operatorname{FibPoly}(Q,P,C(\pi)) \subseteq \operatorname{FibPoly}(Q,P,\Sigma)$ if Σ is a coarsening of $C(\pi)$, which is the case we are interested in.

REMARK 8.42. If Σ is finer than $C(\pi)$, it follows from the equivalence of the descriptions (8.1) and (8.2) of the fiber polytope that $\operatorname{FibPoly}(Q,P,\Sigma) = \operatorname{FibPoly}(Q,P)$.

Another consequence of Lemma 8.41 is the following.

COROLLARY 8.43. *Let Σ_1, Σ_2 be subdivisions of P. If Σ is the common refinement of Σ_1 and Σ_2, then*

$$\operatorname{FibPoly}(Q,P,\Sigma) \subseteq \operatorname{FibPoly}(Q,P,\Sigma_1) \cap \operatorname{FibPoly}(Q,P,\Sigma_2)\,.$$

The relation between fiber polytopes and secondary polytopes is the following.

THEOREM 8.44. [**10**, Theorem 2.5] *Let P be a d-polytope with n vertices and consider the canonical projection $\pi : \Delta(1,n) \to P$. Then*

$$\operatorname{SecPoly}(P) = (d+1) \operatorname{vol} P \cdot \operatorname{FibPoly}(\Delta(1,n),P)\,.$$

In this case, $C(\pi)$ is just the *chamber complex* of P, which is the common refinement of all all regular subdivisions of P. So, given any set $\mathcal{R}$ of regular subdivision of a polytope P, our new construction allows us to give an outer approximation for the secondary polytope by either intersecting all fiber polytopes $\operatorname{FibPoly}(\Delta(1,n),P,\Sigma)$ for all $\Sigma \in \mathcal{R}$ or computing the fiber polytope $\operatorname{FibPoly}(\Delta(1,n),P,\operatorname{CR}(\mathcal{R}))$, where $\operatorname{CR}(\mathcal{R})$ denotes the common refinement of all $\Sigma \in \mathcal{R}$. Note that this construction might be interesting for very different sets of regular subdivisions $\mathcal{R}$. For example, one could take some regular triangulations of P, some very coarse regular subdivision of P, a combination of both, or even a set of regular subdivision constructed in some random way and then study the approximation of $\operatorname{SecPoly}(P)$ arising from our construction.

As splits are the main topic of this thesis, we will take a short look at the case where $\mathcal{R}$ is a set of splits of P.

PROPOSITION 8.45. *Let $\mathcal{S}$ be the set of all splits of a polytope P. Then*

$$(d+1) \operatorname{vol} P \cdot \operatorname{FibPoly}(\Delta(1,n),P,\operatorname{CR}(\mathcal{S})) \subseteq \operatorname{SplitPoly}(P)\,.$$

PROOF. It is easy to see that all the Equations 2.3 are even true for all elements of $(d+1)\operatorname{vol} P \cdot \operatorname{FibPoly}(\Delta(1,n),P,T_P)$, where T_P is the trivial subdivision of P. So, by Lemma 8.41, it suffices to show that for all splits S of P the Inequality (2.6) holds for all $x \in (d+1)\operatorname{vol} P \cdot \operatorname{FibPoly}(\Delta(1,n),P,S)$.

So let $x \in \text{FibPoly}(\Delta(1,n), P, S)$, that is, there exists $y, z \in \mathbb{R}^n_{\geq 0}$ with $x = y + z$ and $V^T y = c_{S_+}$, $V^T z = c_{S_-}$. Suppose that the split hyperplane $H_S = \{x \in \mathbb{R}^{d+1} \mid \langle a, x \rangle = 0\}$ and that $\langle a, x \rangle \geq 0$ for all $x \in S_+$ (so $\langle a, x \rangle \leq 0$ for all $x \in S_-$). Then we compute

$$\begin{aligned}
\sum_{v \in \operatorname{Vert} S_+} |\langle a, v \rangle| x_v &= \sum_{v \in \operatorname{Vert} S_+} \langle a, v \rangle y_v + \underbrace{\sum_{v \in \operatorname{Vert} S_+} \langle a, v \rangle z_v}_{\geq 0} \\
&\geq \sum_{v \in \operatorname{Vert} P} \langle a, v \rangle y_v - \underbrace{\sum_{v \in \operatorname{Vert} S_-} \langle a, v \rangle y_v}_{\leq 0} \\
&\geq a(V^T v) = c_{S_+},
\end{aligned}$$

which shows that (2.6) holds for $(d+1) \operatorname{vol} P \cdot x$. □

It is obvious that, in general, the converse of Proposition 8.45 does not hold since fiber polytopes are trivially bounded, and split polyhedra does not have to be bounded. However, this does not tell us much because we could add the trivial inequalities $x_v \geq 0$ (or even $x_v \geq c$, where $c > 0$ is the smallest volume of any simplex with vertices of P one of which is v) for all Vertices v of P.

Remark 8.46. For the start of a further examination, the following questions that arise naturally from our discussion above, should be answered first:

(a) Does the converse of Corollary 8.43 hold, that is

$$\text{FibPoly}(Q, P, \Sigma) = \text{FibPoly}(Q, P, \Sigma_1) \cap \text{FibPoly}(Q, P, \Sigma_2),$$

where Σ is the common refinement of Σ_1 and Σ_2?

(b) For a split S, do we have equality in

$$\text{FibPoly}(\Delta(1,n), P, S) = \{x \in \Delta(1,n) \mid (d+1) \operatorname{vol} P \cdot x \text{ satisfies (2.6)}\}\,?$$

(c) Is there a sort of generalization of Proposition 8.45 (or the answer to Question (b)) to other facets of the secondary polytope (e.g., k-splits)?

Bibliography

1. Valery Alexeev, *Weighted grassmannians and stable hyperplane arrangements*, 2008, preprint `arXiv:0806.0881`.
2. Federico Ardila and Mike Develin, *Tropical hyperplane arrangements and oriented matroids*, Math. Z. (to appear), preprint `arXiv:0706.2920`.
3. Hans-Jürgen Bandelt and Andreas W. M. Dress, *Reconstructing the shape of a tree from observed dissimilarity data*, Adv. in Appl. Math. **7** (1986), no. 3, 309–343. MR MR858908 (87k:05060)
4. ______, *A canonical decomposition theory for metrics on a finite set*, Adv. Math. **92** (1992), no. 1, 47–105. MR MR1153934 (93h:54022)
5. Robert Bieri and J. R. J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195. MR MR733052 (86c:14001)
6. Louis J. Billera, Richard H. Cushman, and Jan A. Sanders, *The Stanley decomposition of the harmonic oscillator*, Nederl. Akad. Wetensch. Indag. Math. **50** (1988), no. 4, 375–393. MR MR976522 (89m:13013)
7. Louis J. Billera, Israil M. Gel′fand, and Bernd Sturmfels, *Duality and minors of secondary polyhedra*, J. Combin. Theory Ser. B **57** (1993), no. 2, 258–268. MR MR1207491 (93m:52014)
8. Louis J. Billera, Susan P. Holmes, and Karen Vogtmann, *Geometry of the space of phylogenetic trees*, Adv. in Appl. Math. **27** (2001), no. 4, 733–767. MR MR1867931 (2002k:05229)
9. Louis J. Billera, Ning Jia, and Victor Reiner, *A quasisymmetric function for matroids*, 2006, preprint `arXiv:math.CO/0606646`.
10. Louis J. Billera and Bernd Sturmfels, *Fiber polytopes*, Ann. of Math. (2) **135** (1992), no. 3, 527–549. MR MR1166643 (93e:52019)
11. Anders Björner, *Topological methods*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872. MR MR1373690 (96m:52012)
12. Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, second ed., Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1999. MR MR1744046 (2000j:52016)
13. Tristram Bogart, Anders N. Jensen, David E. Speyer, Bernd Sturmfels, and Rekha R. Thomas, *Computing tropical varieties*, J. Symbolic Comput. **42** (2007), no. 1-2, 54–73. MR MR2284285 (2007j:14103)
14. Ethan D. Bolker, *A class of convex bodies*, Trans. Amer. Math. Soc. **145** (1969), 323–345. MR MR0256265 (41 #921)
15. Peter Buneman, *A note on the metric properties of trees*, J. Combinatorial Theory Ser. B **17** (1974), 48–50. MR MR0363963 (51 #218)
16. Marek Chrobak and Lawrence L. Larmore, *A new approach to the server problem*, SIAM J. Discrete Math. **4** (1991), no. 3, 323–328. MR MR1105939 (92b:90090)
17. ______, *Generosity helps or an* 11*-competitive algorithm for three servers*, J. Algorithms **16** (1994), no. 2, 234–263. MR MR1258238 (94m:90046)
18. David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992, An introduction to computational algebraic geometry and commutative algebra. MR MR1189133 (93j:13031)
19. Jesús A. De Loera, Jörg Rambau, and Francisco Santos, *Triangulations: Structures and algorithms*, Springer, to appear.

20. Jesús A. De Loera, Bernd Sturmfels, and Rekha R. Thomas, *Gröbner bases and triangulations of the second hypersimplex*, Combinatorica **15** (1995), no. 3, 409–424. MR MR1357285 (97b:13035)
21. Mike Develin and Bernd Sturmfels, *Tropical convexity*, Doc. Math. **9** (2004), 1–27 (electronic), correction: ibid., pp. 205–206. MR MR2054977 (2005i:52010)
22. Andreas Dress, Stefan Grünewald, Jakob Jonsson, and Vincent Moulton, *The simplicial complex $\delta_{n,k}$ of k compatible line arrangements in the hyperbolic plane - part 1: The structure of $\delta_{n,k}$*, 2007.
23. Andreas W. M. Dress, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces*, Adv. in Math. **53** (1984), no. 3, 321–402. MR MR753872 (86j:05053)
24. Andreas W. M. Dress, K. Huber, and Vincent Moulton, *Some variations on a theme by Buneman*, Ann. Comb. **1** (1997), no. 4, 339–352. MR MR1630743 (99h:05114b)
25. Andreas W. M. Dress, Katharina T. Huber, Jacobus H. Koolen, and Vincent Moulton, *Six points suffice: how to check for metric consistency*, European J. Combin. **22** (2001), no. 4, 465–474. MR MR1829740 (2002b:05036)
26. Andreas W. M. Dress, Katharina T. Huber, and Vincent Moulton, *An exceptional split geometry*, Ann. Comb. **4** (2000), no. 1, 1–11. MR MR1763946 (2001h:92005)
27. ———, *An explicit computation of the injective hull of certain finite metric spaces in terms of their associated Buneman complex*, Adv. Math. **168** (2002), no. 1, 1–28. MR MR1907317 (2003g:54077)
28. Andreas W. M. Dress and Daniel H. Huson, *Constructing splits graphs*, IEEE/ACM Trans. Comput. Biol. Bioinformatics **1** (2004), no. 3, 109–115.
29. Andreas W. M. Dress and Walter Wenzel, *Perfect matroids*, Adv. Math. **91** (1992), no. 2, 158–208. MR MR1149622 (93a:05046)
30. ———, *Valuated matroids*, Adv. Math. **93** (1992), no. 2, 214–250. MR MR1164708 (93h:05045)
31. Jean-Guillaume Dumas, Frank Heckenbach, David Saunders, and Volkmar Welker, *Simplicial homology, a (proposed) gap share package*, 2008, `http://www.cis.udel.edu/~dumas/Homology/`.
32. Eugène Ehrhart, *Fraction génératrice d'un système diophantien linéaire homothétique*, C. R. Acad. Sci. Paris **252** (1961), 651–653. MR MR0122782 (23 #A121a)
33. ———, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618. MR MR0130860 (24 #A714)
34. ———, *Polynômes arithmétiques et méthode des polyèdres en combinatoire*, Institut de Recherche Mathématique Avancée, Strasbourg, 1974. MR MR0345913 (49 #10642)
35. Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005), no. 4, 437–468. MR MR2191630 (2006j:05036)
36. Komei Fukuda, *Cddlib reference manual*, 2005, cddlib version 94, `http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html`.
37. Ewgenij Gawrilow and Michael Joswig, `polymake`*: a framework for analyzing convex polytopes*, Polytopes–combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR 2001f:52033
38. Israil M. Gel′fand, Mark Goresky, Robert D. MacPherson, and Vera V. Serganova, *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. **63** (1987), no. 3, 301–316. MR MR877789 (88f:14045)
39. Israil M. Gel′fand, Mikhail M. Kapranov, and Andrey V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994. MR MR1264417 (95e:14045)
40. Israil M. Gel′fand, Andrey V. Zelevinskiy, and Mikhail M. Kapranov, *Discriminants of polynomials in several variables and triangulations of Newton polyhedra*, Algebra i Analiz **2** (1990), no. 3, 1–62. MR MR1073208 (91m:14080)
41. Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.
42. Branko Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee, and Günter M. Ziegler. MR MR1976856 (2004b:52001)

43. Paul Hacking, *The homology of tropical varieties*, Collectanea Mathematica **59** (2008), 263–273.
44. Paul Hacking, Sean Keel, and Jenia Tevelev, *Compactification of the moduli space of hyperplane arrangements*, J. Algebraic Geom. **15** (2006), no. 4, 657–680. MR MR2237265 (2007j:14016)
45. Sven Herrmann, Anders N. Jensen, Michael Joswig, and Bernd Sturmfels, *How to draw tropical planes*, (2008), preprint `arXiv:0808.2383`.
46. Sven Herrmann and Michael Joswig, *Bounds on the f-vectors of tight spans*, Contrib. Discrete Math. **2** (2007), no. 2, 161–184 (electronic). MR MR2358269
47. Sven Herrmann and Michael Joswig, *Splitting polytopes*, Münster J. Math. (2008), no. 1, 109–142.
48. ———, *Totally splittable polytopes*, (2009), preprint `arXiv:0901.0231`.
49. Hiroshi Hirai, *Characterization of the distance between subtrees of a tree by the associated tight span*, Ann. Comb. **10** (2006), no. 1, 111–128. MR MR2233884 (2007f:05058)
50. ———, *A geometric study of the split decomposition*, Discrete Comput. Geome. **36** (2006), no. 2, 331–361.
51. Katharina T. Huber, Jacobus H. Koolen, and Vincent Moulton, *The tight span of an antipodal metric space. II. Geometrical properties*, Discrete Comput. Geom. **31** (2004), no. 4, 567–586. MR MR2053499 (2005f:54055)
52. ———, *On the structure of the tight-span of a totally split-decomposable metric*, European J. Combin. **27** (2006), no. 3, 461–479. MR MR2206480 (2006k:52040)
53. Peter Huggins, Bernd Sturmfels, Josephine Yu, and Debbie S. Yuster, *The hyperdeterminant and triangulations of the 4-cube*, Math. Comp. **77** (2008), 1653–1679.
54. Daniel H. Huson and David Bryant, *Application of phylogenetic networks in evolutionary studies*, Mol. Biol. Evol. **23** (2006), no. 2, 254–267.
55. John R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76. MR MR0182949 (32 #431)
56. Anders N. Jensen, *A presentation of the Gfan software*, Mathematical software—ICMS 2006, Lecture Notes in Comput. Sci., vol. 4151, Springer, Berlin, 2006, pp. 222–224. MR MR2387172
57. Jakob Jonsson, *Generalized triangulations and diagonal-free subsets of stack polyominoes*, J. Combin. Theory Ser. A **112** (2005), no. 1, 117–142. MR MR2167478 (2006d:05011)
58. Michael Joswig, *Projectivities in simplicial complexes and colorings of simple polytopes*, Math. Z. **240** (2002), no. 2, 243–259. MR MR1900311 (2003f:05047)
59. Michael Joswig and Katja Kulas, *Tropical and ordinary convexity combined*, 2008, preprint `arXiv:0801.4835`, to appear in Adv. Geometry.
60. Mikhail M. Kapranov, *Chow quotients of Grassmannians. I*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110. MR MR1237834 (95g:14053)
61. Sean Keel and Jenia Tevelev, *Geometry of Chow quotients of Grassmannians*, Duke Math. J. **134** (2006), no. 2, 259–311. MR MR2248832 (2007m:14017)
62. Sangwook Kim, *Flag enumeration of matroid base polytopes*, 2009, preprint `arXiv:0801.4835`.
63. Shungo Koichi, *The buneman index via polyhedral split decomposition*, 2006, METR 2006-57.
64. Jacobus H. Koolen and Vincent Moulton, *A note on the uniqueness of coherent decompositions*, Adv. in Appl. Math. **19** (1997), no. 4, 444–449. MR MR1479012 (98j:54050)
65. Jacobus H. Koolen, Vincent Moulton, and Udo Tönges, *The coherency index*, Discrete Math. **192** (1998), no. 1-3, 205–222, Discrete metric spaces (Villeurbanne, 1996). MR MR1656733 (99g:54027)
66. Martin Kreuzer and Lorenzo Robbiano, *Computational commutative algebra. 2*, Springer-Verlag, Berlin, 2005. MR MR2159476 (2006h:13036)
67. Thomas Lam and Alexander Postnikov, *Alcoved polytopes. I*, Discrete Comput. Geom. **38** (2007), no. 3, 453–478. MR MR2352704 (2008m:52031)
68. Carl W. Lee, *Subdivisions and triangulations of polytopes*, Handbook of discrete and computational geometry, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997, pp. 271–290. MR MR1730170
69. Hannah Markwig and Josephine Yu, *The space of tropically collinear points is shellable*, Collect. Math. **60** (2009), no. 1, 63–77.

70. Grigory Mikhalkin, *Introduction to tropical geometry*, 2007, IMPA notes and book manuscript, preprint `arXiv:0709.104`.
71. Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR MR2110098 (2006d:13001)
72. Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992. MR MR1217488 (94e:52014)
73. Lior Pachter and Bernd Sturmfels (eds.), *Algebraic statistics for computational biology*, Cambridge University Press, New York, 2005. MR MR2205865 (2006i:92002)
74. Sam Payne, *Ehrhart series and lattice triangulations*, Discrete Comput. Geom. **40** (2008), no. 3, 365–376. MR MR2443289
75. Julian Pfeifle, *The secondary polytope of the* 3*-cube*, Electronic Geometry Models (2000), `www.eg-models.de/2000.09.031`.
76. Vincent Pilaud and Francisco Santos, *Multi-triangulations as complexes of star polygons*, Discrete Comput. Geom. **41** (2009), no. 2, 284–317.
77. Konrad Polthier, Klaus Hildebrandt, Eike Preuss, and Ulrich Reitebuch, *JavaView, version 3.95*, `http://www.javaview.de`, 2007.
78. Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald, *First steps in tropical geometry*, Idempotent mathematics and mathematical physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 289–317. MR MR2149011 (2006d:14073)
79. Alan Robinson and Sarah Whitehouse, *The tree representation of* Σ_{n+1}, J. Pure Appl. Algebra **111** (1996), no. 1-3, 245–253. MR MR1394355 (97g:55010)
80. Francisco Santos, *The Cayley trick and triangulations of products of simplices*, Integer points in polyhedra—geometry, number theory, algebra, optimization, Contemp. Math., vol. 374, Amer. Math. Soc., Providence, RI, 2005, pp. 151–177. MR MR2134766 (2005k:52025)
81. David E Speyer, *Tropical geometry*, Ph.D. thesis, UC Berkeley, 2005.
82. ———, *A matroid invariant via k-theory of the grassmannian*, 2006, preprint `arXiv:math/0603551`, to appear in Adv. in Math.
83. David E. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558. MR MR2448909
84. David E Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411. MR MR2071813 (2005d:14089)
85. David E Speyer and Lauren Williams, *The tropical totally positive Grassmannian*, J. Algebraic Combin. **22** (2005), no. 2, 189–210. MR MR2164397 (2006e:14084)
86. Richard P. Stanley, *Eulerian partitions of the unit hypercube*, p. 49, D. Reidel, 1977.
87. ———, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342, Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978). MR MR593545 (82a:52007)
88. Bernd Sturmfels, *Gröbner bases of toric varieties*, Tohoku Math. J. (2) **43** (1991), no. 2, 249–261. MR MR1104431 (92j:14067)
89. ———, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR MR1363949 (97b:13034)
90. Bernd Sturmfels and Nobuki Takayama, *Gröbner bases and hypergeometric functions*, Gröbner bases and applications (Linz, 1998), London Math. Soc. Lecture Note Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998, pp. 246–258. MR MR1708882 (2001c:33026)
91. Bernd Sturmfels and Josephine Yu, *Classification of six-point metrics*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 44, 16 S. MR MR2097310
92. Henryk Trappmann and Günter M. Ziegler, *Shellability of complexes of trees*, J. Combin. Theory Ser. A **82** (1998), no. 2, 168–178. MR MR1620865 (99f:05122)
93. Karen Vogtmann, *Local structure of some* $\mathrm{Out}(F_n)$*-complexes*, Proc. Edinburgh Math. Soc. (2) **33** (1990), no. 3, 367–379. MR MR1077791 (92d:57002)

94. Neil White (ed.), *Theory of matroids*, Encyclopedia of Mathematics and its Applications, vol. 26, Cambridge University Press, Cambridge, 1986. MR MR849389 (87k:05054)
95. Neil White (ed.), *Matroid applications*, Encyclopedia of Mathematics and its Applications, vol. 40, Cambridge University Press, Cambridge, 1992. MR MR1165537 (92m:05004)
96. Roland Wunderling, *Paralleler und objektorientierter Simplex-Algorithmus*, Ph.D. thesis, Technische Universität Berlin, 1996.
97. Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR MR1311028 (96a:52011)

Wissenschaftlicher Werdegang

17. April 1982	geboren in Darmstadt.
06/2001	Abitur an der Alfred-Delp-Schule in Dieburg
04/2002–10/2005	Diplomstudium der Mathematik an der TU Darmstadt
11/2005–03/2007	wissenschaftlicher Mitarbeiter und Doktorand der Mathematik an der TU Darmstadt
04/2007–03/2009	Stipendiat und Doktorand der Mathematik an der TU Darmstadt